UDC 510.53
МРНТИ 27.03.45 DOI 10.55452/1998-6688-2021-18-2-53-58

INFINITE FAMILIES OF TOTAL FUNCTIONS WITH PRINCIPAL NUMBERINGS

ISSAKHOV A.A., RAKYMZHANKYZY F., OSTEMIROVA U.
Kazakh-British technical university, 050000, Almaty, Kazakhstan

Abstract. *It was known that any non-single-element (in particular, any infinite) family of total functions with an oracle A, such that $\emptyset' \leq_T A$, does not have A-computable principal numbering; later it was proved that any finite family of total functions with a hyperimmune-free oracle A always has an A-computable principal numbering. The unresolved question was whether there exists an infinite family of total functions with a hyperimmune-free oracle A that has an A-computable principal numbering. The paper gives a positive answer to this question: it is proved that there exists an infinite A-computable family F of total functions, where the Turing degree of the set A is hyperimmune-free, such that F has an A-computable principal numbering.*

Key words: *A-computable numbering, hyperimmune oracle, hyperimmune-free oracle, principal numbering.*

БАСТЫ НӨМІРЛЕУЛЕРІ БАР БАРЛЫҚ ЖЕРДЕ АНЫҚТАЛҒАН ФУНКЦИЯЛАРДАН ТҰРАТЫН ШЕКСІЗ ҮЙІРЛЕР

ИСАХОВ А.А., РАХЫМЖАНҚЫЗЫ Ф., ӨСТЕМІРОВА У.
Қазақстан-Британ техникалық университеті, 050000, Алматы, Қазақстан

Аңдатпа. *Кез келген бір-элементті емес (жеке жағдайда, кез келген шексіз) барлық жерде анықталған функциялардан тұратын A оракулымен есептелетін үйірлердің A-есептелімді бас нөмірлеуі болмайтыны белгілі болған және бұл тұжырым $\emptyset' \leq_T A$ үшін дұрыс; кейінірек, кез келген барлық жерде анықталған функциялардан тұратын гипериммунды-бос A оракулымен есептелетін* **This work was supported by the Science Committee of the Republic of Kazakhstan (Grant AP08856493)* *ақырлы үйірлердің әрдайым A-есептелімді бас нөмірлеуі болатыны дәлелденген. A-есептелімді бас нөмірлеуі бар барлық жерде анықталған функциялардан тұратын гипериммунды-бос A оракулымен есептелетін ақырсыз үйір табылатыны шешілмеген мәселе болып қалған. Осы жұмыста бұл мәселе оң шешімін тапты: A-есептелімді бас нөмірлеуі бар барлық жерде анықталған функциялардан тұратын гипериммунды-бос A оракулымен есептелетін ақырсыз үйір табылатыны дәлелденген.*

Түйінді сөздер: *A-есептелімді нөмірлеу, гипериммунды оракул, гипериммунды-бос оракул, бас нөмірлеу.*

БЕСКОНЕЧНЫЕ СЕМЕЙСТВА ВСЮДУ ОПРЕДЕЛЕННЫХ ФУНКЦИЙ С ГЛАВНЫМИ НУМЕРАЦИЯМИ

ИСАХОВ А.А., РАХЫМЖАНКЫЗЫ Ф., ОСТЕМИРОВА У.
Казахстанско-Британский технический университет, 050000, Алматы, Казахстан

Аннотация. *Ранее было известно, что любое не одноэлементное (в частности, любое бесконечное) семейство всюду определенных функций с оракулом A, такое что $\emptyset' \leq_T A$, не имеет A-вычислимую главную нумерацию, позже было доказано, что любое конечное семейство всюду определенных функций с гипериммунно-свободным оракулом A всегда обладает A-вычислимой главной нумерацией. Оставался нерешенным вопрос о том, существует ли бесконечное семейство всюду определенных функций с* *гипериммунно-свободным оракулом **A**, которое имеет -вычислимую главную нумерацию. В работе приводится положительный ответ на указанный вопрос: доказано, что существует бесконечное -вычислимое семейство **F** всюду определенных функций, где тьюрингова степень множества **A** гипериммунно-свободна, такое, что **F** имеет **A**-вычислимую главную нумерацию.*

Ключевые слова: A*-вычислимая нумерация, гипериммунный оракул, гипериммунно-свободный оракул, главная нумерация.*

Introduction

***A*-computable principal numberings**

All basic definitions related to computable numberings can be found in [1].

If a family F of total functions is A-computable for arbitrary set A, then a numbering α of the family F is called A-*computable* while $\alpha(n)(x)$ is A-computable binary function, [2]. If degree $\boldsymbol{a}$ contains a hyperimmune set, then $\boldsymbol{a}$ is *hyperimmune*. Otherwise, $\boldsymbol{a}$ is *hyperimmune-free*.

It is known that if an A-computable family F of total functions contains at least two elements, where A is a hyperimmune set (for $A \geq \emptyset'$ see [3]), then F has no A-computable principal numbering, [4]. It was also proved that if finite A-computable family F of total functions, where Turing degree of the set A is hyperimmune-free, then F has an A-computable principal numbering, [5].

Now we introduce alternative proofs of some theorems related to the main theorem.

Main part

Theorem (Issakhov, [3]). If an A-computable family F contains at least two functions, where $\emptyset' \leq_T A$, then F has no A-computable principal numbering.

Proof. Let α be an A-computable numbering of the family F. f and g are two different functions from F. Therefore there exists $a \in \omega$ such that

$$f(a) \neq g(a).$$

Construct a numbering β such that for any $e \in \omega$:

$$\beta(e) \neq \alpha\varphi_e(e).$$

Let

$$K = \{a_0 < a_1 < a_2 < \cdots\},$$

$$\overline{K} = \{b_0 < b_1 < b_2 < \cdots\}$$

be halting set and its complement, respectively, which are A-computable sets. Let's define

$$\beta(b_i) = \alpha(i)$$

for any $i \in \omega$, which guarantee that β is an A-computable numbering of the family F. But

$$\beta(a_i) = \begin{cases} f, if\ \alpha\varphi_{a_i}(a_i)(a) = g(a) \\ g, \quad otherwise \end{cases}$$

Then β is still A-computable numbering of the family F, and

$$\beta(a_i)(a) \neq \alpha\varphi_{a_i}(a_i)(a).$$

Therefore, for any α there exists β such that $\beta \nleq \alpha$, which means α can not be a principal numbering of the family F.

Theorem is proved.

Note that every finite family of computably enumerable (c.e.) sets has a principal computable numbering.

Theorem (Badaev, Goncharov, [2]). For every set A, where $\emptyset' \leq_T A$, a finite family S of A-c.e. sets has an A-computable principal numbering if and only if S contains the least set under inclusion.

Theorem (Badaev, Goncharov, [2]). For every set A, there is an infinite A-computable family S of sets with pairwise disjoint elements such that S has an A-computable principal numbering.

Compare with the results in [6].

An infinite set A is hyperimmune if and only if no recursive function majorizes A, [7].

Theorem (Miller, Martin), [8].

(a) If $\boldsymbol{a}$ is hyperimmune and $\boldsymbol{a} < \boldsymbol{b}$ then $\boldsymbol{b}$ is hyperimmune.

(b) $\boldsymbol{a}$ is hyperimmune iff some function of degree $\leq \boldsymbol{a}$ is majorized by no recursive function.

(c) If $(\exists \boldsymbol{a})[\boldsymbol{a} < \boldsymbol{b} < \boldsymbol{a}']$, then $\boldsymbol{b}$ is hyperimmune.

(d) Every nonzero degree comparable with $\mathbf{0}'$ is hyperimmune.

Theorem (Issakhov, [5]). Let A be a hyperimmune set. If A-computable family F of total functions contains at least two elements, then F has no principal A-computable numbering.

Proof. Let α be an A-computable numbering of the family F, and f and g be two different functions from F. Then there exists a number a such that

$$f(a) \neq g(a).$$

Construct β such that $\beta \nleq \alpha$, i.e. for any $e \in \omega$: $\beta \neq \alpha\varphi_e$. Let

$$\beta(\langle 0,e,x\rangle) = \begin{cases} g, if\ \alpha\varphi_{e,h(x)}(\langle 0,e,x\rangle)(a) = f(a) \\ f, \quad otherwise \end{cases}$$

where h is a non-majorized A-computable function. And

$$\beta(\langle n,e,x\rangle) = \alpha(\langle e,x\rangle)$$

for all $n > 0$, i.e. β is the numbering of the family F.

If φ_e is not total, then

$$\beta(x) \neq \alpha\varphi_e(x)$$

for some $x \in \omega$.

If φ_e is total, then

$$\tilde{h}(x) = \min\{s:\ \varphi_{e,s}(\langle 0,e,x\rangle)\downarrow\}$$

is computable function.

Since h is a non-majorized A-computable function, then there exists a number $b \in \omega$ such that $\tilde{h}(b) < h(b)$. Therefore

$$\varphi_{e,h(b)}(\langle 0,e,b\rangle)\downarrow,$$

and

$$\beta(\langle 0,e,b\rangle) \neq \alpha\varphi_e(\langle 0,e,b\rangle).$$

Theorem is proved.

Theorem (Issakhov, [5]). If Turing degree of a set A is hyperimmune-free then every A-computable finite family of total functions has an A-computable principal numbering.

Proof. Note that a degree $\boldsymbol{a}$ is hyperimmune if and only if some function of degree $\leq \boldsymbol{a}$ is majorized by no recursive function. Therefore a degree $\boldsymbol{a}$ is hyperimmune-free if and only if for any function f of degree $\leq \boldsymbol{a}$ there exists computable function g such that $f(n) \leq g(n)$ for all $n \in \omega$.

Let α be an A-computable numbering of the finite family

$$F = \{f_0, f_1, \dots, f_n\}$$

of total functions. Define distinguishing sets $A_0, A_1, \dots, A_n$ of the functions $f_0, f_1, \dots, f_n$ which satisfy the next two conditions:

(1) $A_i = \{a_{i0}, a_{i1}, \dots, a_{im}\}$ for some $m \leq n$,

(2) for any $j \leq n$ there exists $k \leq m$ such that

$$\text{if } j \neq i \text{ then } f_i(a_{ik}) \neq f_j(a_{ik}).$$

It means

$$A_0 = \{a_{00}, a_{01}, \dots, a_{0m_0} | f_0(a_{00}) \neq f_1(a_{00}), f_0(a_{01}) \neq f_2(a_{01}), \dots, f_0(a_{0m_0}) \neq f_n(a_{0m_0})\}$$

$$A_1 = \{a_{10}, a_{11}, \dots, a_{1m_1} | f_1(a_{10}) \neq f_0(a_{10}), f_1(a_{11}) \neq f_2(a_{11}), \dots, f_1(a_{1m_1}) \neq f_n(a_{1m_1})\}$$

… …

$$A_n = \{a_{n0}, a_{n1}, \dots, a_{nm_n} | f_n(a_{n0}) \neq f_0(a_{n0}), f_n(a_{n1}) \neq f_1(a_{n1}), \dots, f_n(a_{nm}) \neq f_{n-1}(a_{nm_n})\}$$

where $m_0, m_1, \dots, m_n \leq n$.

Denote by $\Phi_e^A(x,y)$ all A-computable binary functions by $e \in \omega$. Then there exists e such that

$$\Phi_e^A(x)(y) = \Phi_e^A(x,y) \equiv \alpha(x)(y),$$

and the same for any numbering. Let

$$\beta(\langle e,x,s\rangle)(y)=\begin{cases} f_i(y), & if\ \exists i\leq n\ \left(f_i(a_{ik})=\Phi^A_{e,s}(x)(a_{ik})\right)\ for\ all\ k\leq |A_i| \\ f_0(y), & otherwise\end{cases}$$

From construction it is easy to see that β is A-computable numbering of the family F. We show that β is A-computable principal numbering of F.

Let γ be an A-computable numbering of the family F. Then there exists $e_1\in\omega$ such that

$$\gamma(x)=\Phi^A_{e_1}(x)$$

for any $x\in\omega$. Let

$$g(x)=\min\left\{s:\ \exists i\leq n\ \left(f_i(a_{ik})=\right.\right.$$

$$\left.\left.=\Phi^A_{e_1,s}(x)(a_{ik})\right)\ for\ all\ k\leq|A_i|\right\}.$$

$g(x)$ is an A-computable total function, where Turing degree of the set A is hyperimmune-free. Therefore, there exists a computable function f such that $g(x)\leq f(x)$ for any $x\in\omega$. Then

$$\beta(\langle e_1,x,f(x)\rangle)(y)=f_i(y)=$$
$$=f_i(y)=\Phi^A_{e_1}(x)(y)=\gamma(x)(y)$$

for all $y\in\omega$, i.e.

$$\beta(\langle e_1,x,f(x)\rangle)=\gamma(x)$$

which means $\gamma\leq\beta$.

Theorem is proved.

Note that the Rogers semilattice of the family from previous theorem is not trivial, since

Theorem (Issakhov, [5]). Let $\emptyset<_T A$. Then any A-computable non-trivial finite family of total functions has at least two non-equivalent A-computable numberings.

Proof. Let

$$F=\{f_0,f_1,\dots,f_n\},$$

where $n>0$,

$$\alpha(i)=\begin{cases} f_i, & if\ i\leq n\\ f_0, & if\ i>n\end{cases}$$

and

$$\beta(i)=\begin{cases} f_i, & if\ i\leq n\\ f_0, & if\ i>n\ and\ i\in A\\ f_1, & if\ i>n\ and\ i\notin A\end{cases}$$

We show that $\alpha\not\equiv\beta$. Assume $\alpha\equiv\beta$, then there exists a computable function g such that $\beta(x)=\alpha(g(x))$. Note that

$$x\in A\backslash[0;n]\Leftrightarrow(x>n)\ and\ (g(x)=0\ or\ g(x)>n)$$

Since the right part is computable then $A\backslash[0;n]$ is also computable. Therefore A is computable too, contradiction.

Theorem is proved.

Main result

Theorem. There exists an infinite A-computable family F of total functions, where Turing degree of the set A is hyperimmune-free, such that F has an A-computable principal numbering.

Proof. Let's remind that $\boldsymbol{a}$ is hyperimmune-free degree if and only if for any $f\leq_T\boldsymbol{a}$ there exists computable function g such that

$$f(n)\leq g(n)$$

for all $n\in\omega$.

Assume that the infinite A-computable family F of total functions consists of only constant functions $f_0,f_1,\dots,f_n,\dots$ where

$$f_i(x)=i$$

for all $x\in\omega$. Denote by $\Phi^A_e(x,y)$ all A-computable binary functions by $e\in\omega$. Let α be an arbitrary A-computable numbering of the family F. Then there exists a number e such that

$$\alpha(x)(y)=\Phi^A_e(x,y).$$

Let

$$\beta(\langle e,x,s\rangle)(y)=\begin{cases} f_i(y), & if\ \exists i\left(f_i(k)=\right.\\ f_0(y),\end{cases}$$

$= \Phi^A_{e,s}(x)(k)\big)$ $for\ all\ k \le s$

$otherwise.$

From construction of β it is easy to see that β is an A-computable numbering of the family F. Let's show that β is a principal numbering of the family F.

Take any numbering γ of the family F and show that $\gamma \le \beta$. Since γ is a numbering of the family F, there exists a number $e_1 \in \omega$ such that

$\gamma(x) = \Phi^A_{e_1}(x)$

for any $x \in \omega$.

$g(x) = \min\left\{s : \exists i \left(f_i(k) = \Phi^A_{e_1,s}(x)(k)\right) for\ all\ k \le s\right\}$

g is an A-computable total function, where Turing degree of the set A is hyperimmune-free. Therefore, there exists computable function f such that $g(x) \le f(x)$ for any $x \in \omega$. It means that

$\beta(\langle e_1, x, f(x)\rangle)(y) = f_i(y) = \Phi^A_{e_1,s}(x)(y) = \gamma(x)(y)$

for all $y \in \omega$, i.e.

$\gamma(x) = \beta(\langle e_1, x, f(x)\rangle).$

Therefore $\gamma \le \beta$.

Theorem is proved.

Conclusion

Note that, [4, 9], if an A-computable family F of total functions contains at least two elements, where A is a hyperimmune set, then F has no A-computable principal numbering; but if Turing degree of the oracle set A is hyperimmune free then any finite A-computable family F of total functions has an A-computable principal numbering, and also now there exists an infinite A-computable family F of total functions such that F has an A-computable principal numbering.

Question. Is there an infinite A-computable family of total functions, where Turing degree of the set A is hyperimmune free, with no A-computable principal numbering?

REFERENCES

1. Yu.L. Ershov. Theory of numberings \\ Handbook of Computability Theory. – North-Holland; Amsterdam: Stud. Log. Found. Math., 1999, Vol. 140, pp. 473-503.
2. S.A. Badaev and S.S. Goncharov, Generalized computable universal numberings, Algebra and Logic, vol. 53 (2014), no. 5, pp. 355-364.
3. A.A. Issakhov, Ideals without minimal elements in Rogers semilattices, Algebra and Logic, vol. 54 (2015), no. 3, pp. 197-203.
4. A.A. Issakhov, A-computable numberings of the families of total functions, The Bulletin of Symbolic Logic, vol. 22 (2016), no. 3, p. 402.
5. Assylbek Issakhov, Hyperimmunity and A-computable universal numberings, AIP Conference Proceedings, vol. 1759, 020106 (2016); doi: 10.1063/1.4959720.
6. M.Kh. Faizrakhmanov, Universal generalized computable numberings and hyperimmunity, Algebra and Logic, vol. 56 (2017), no. 4, pp. 337-347.
7. Soare R.I., Recursively enumerable sets and degrees. – Berlin; Heidelberg; New York: Springer-Verlag, 1987. – 437 p.
8. Miller W., Martin D.A., The degree of hyperimmune sets \\ Z. Math. Logik Grundlag. Math., 1968, Vol. 14, pp. 159-166.
9. Issakhov A.A., Rakymzhankyzy F., Hyperimmunity and A-computable numberings \\ The Bulletin of Symbolic Logic, 2018, Vol. 24, No. 2, pp. 248-249.

Information about authors:

1. Assylbek A. Issakhov – PhD Doctor, Head and Professor of the Scientific and Educational Center of Mathematics and Cybernetics KBTU, Kazakh-British Technical University, st. Tole bi 59
ORCID ID: https://orcid.org/0000-0001-7020-7988 Email: a.isakhov@kbtu.kz
2. Fariza Rakymzhankyzy – Master of Science, PhD student, Scientific and Educational Center for Mathematics and Cybernetics, Kazakh-British Technical University, st. Tole bi 59
ORCID ID: https://orcid.org/0000-0002-6517-5560
Email: fariza.rakymzhankyzy@gmail.com
3. Uldana B. Ostemirova – PhD student, tutor of the Scientific and Educational Center of Mathematics and Cybernetics, Kazakh-British Technical University, st. Tole bi 59
Email: u.ostemirova@kbtu.kz