UDC 517.3
IRSTI 27.41.19

https://doi.org/10.55452/1998-6688-2023-20-3-38-44

Beisenbay A. A.

Institute of Mathematics and Mathematical Modeling, 050010, Almaty, Kazakhstan
Al-Farabi Kazakh National University 050040, Almaty, Kazakhstan
E-mail: beisenbay@math.kz

ON AN ANALOGUES OF VAN DER CORPUT LEMMAS

Abstract. The article deals with analogues of the van der Corput lemmas involving Mittag-Leffler functions. In fact, the classical estimates for the oscillatory integrals was obtained by the Dutch mathematician Johannes Gaultherus van der Corput and named in his honour. Various generalisations of the van der Corput lemmas have been investigated over the years. The general statement is that we replace the exponential function with the Mittag-Leffler-type function, to study oscillatory integrals appearing in the analysis of time-fractional partial differential equations. For a particular purpose, the present paper focuses on the integral of the form

$$I_{\alpha,\beta}(\lambda) = \int_R E_{\alpha,\beta}(\zeta\lambda\phi(x))\psi(x)dx,$$

in the case β = α, for the range 0 < α < 2. This make wider the variety of estimates obtained in the work [1], where integrals with functions $E_{\alpha,\beta}(i^\alpha\lambda\phi(x))$ have been studied. In this paper we interested in the behavior of the integral $I_{\alpha,\beta}(\lambda)$ when λ is large. We realize that the decay rates of the integral which, we have been considered, depend on the ranges of parameters α and β. Further studying Van der Corput lemmas will reveal new insights and applications in mathematics and beyond.

Key words: Van Der Corput lemmas, Mittag-Leffler function, inductive hypothesis, estimate, smooth function, oscillatory integral

Introduction

In this paper we will expand the study of oscillatory-type integrals involving Mittag-Leffler functions $E_{\alpha,\beta}$ initiated in [1]. In the case of $\alpha = \beta = 1$, we have $E_{1,1}(z) = e^z$, thus reducing the integral to the classical question of decay of oscillatory integrals. Indeed, the estimate obtained by the Dutch mathematician Johannes Gaultherus van der Corput [2] and named in his honour, following Stein [3], can be stated as follows:

Van der Corput lemma. Suppose ϕ is a real-valued and smooth function in [a;b]: If ψ is a smooth function and $\phi^{(k)}(x) \vee\geq 1, k \geq 1$ for all $x \in (a,b)$, then

$$\left|\int_a^b e^{-i\lambda\phi(x)}\psi(x)dx\right| \leq C\lambda^{-1/k}, \lambda \longrightarrow \infty, \tag{1.1}$$

for k = 1 and ϕ' is monotonic, or k ≥ 2. Here C does not depend on λ.

Various generalisations of the van der Corput lemmas have been investigated over the years [3], [4-10]. Multidimensional analogues of the van der Corput lemmas were studied in [11-17], while in [18] the multi-dimensional van der Corput lemma was obtained with constants independent of the phase and amplitude.

We consider the generalized oscillatory integral defined by

$$I_{\alpha,\beta}(\lambda) = \int_R E_{\alpha,\beta}(\zeta\lambda\phi(x))\psi(x)dx, \tag{1.2}$$

where $0 < \alpha < 2, \beta > 0, \zeta \in C, \phi$ is a phase and ψ is an amplitude, and λ –is a positive real number that can vary. Here $E_{\alpha,\beta}(\zeta\lambda\phi(x))$ is the Mittag-Leffler function defined as (see e.g. [19], [20])

$$E_{\alpha,\beta}(\zeta\lambda\phi(x)) = \sum_{k=0}^{\infty}\frac{(\zeta\lambda\phi(x))^k}{\Gamma(\alpha k+\beta)}, \alpha > 0, \beta \in R,$$

with the property that

$$E_{1,1}(\zeta\lambda\phi(x)) = e^{\zeta\lambda\phi(x)}. \quad (1.3)$$

Since the function $E_{\alpha,\beta}(\zeta\lambda\phi(x))$ has a set of real zeros [20], the integral (1.2) is oscillating.

The main goal of the present paper is to obtain van der Corput-type estimates for the oscillatory integral (1.2) in the form

$$\left|\int_a^b E_{\alpha,\alpha}(\zeta\lambda\phi(x))\psi(x)dx\right| \le C\lambda^{\frac{-1}{k}}, \lambda \longrightarrow \infty. \quad (1.4)$$

It is one extension of (1.1) in view of (1.3), namely, an extension (in Theorem 3.5) to the range $0 < \alpha < 2$, , for $k = 1$ and ϕ' is monotonic, or $k \ge 2$.

This present paper is a generalization of [1], where a variety of van der Corput type lemmas were obtained for the integral defined by

$$\mathcal{I}_{\alpha,\beta}(\lambda) = \int_R E_{\alpha,\beta}(i^\alpha\lambda\phi(x))\psi(x)dx. \quad (1.5)$$

As we see above, the integral (1.5) is different from the integral (1.2), since in (1.5) there is a purely imaginary number i^α before the phase function, and in (1.2) the complex number, i.e. ζ. As in the case of (1.5) studied in [1], we find that the decay rates of (1.2) when, $\beta = \alpha$, as $\lambda \longrightarrow \infty$ depend on the ranges of parameters and . We also obtain more results in the case of in finite intervals.

Main provisions. Material and methods

Proposition 2.1 ([Pod99]). If $0 < \alpha < 2$, β is an arbitrary real number, μ is such that $\pi\alpha/2 < \mu < min\{\pi, \pi\alpha\}$, then there is $C_1, C_2 > 0$, such that we have

$$|E_{\alpha,\beta}(\zeta\lambda\phi(x))| \le C_1(1 + |\zeta\lambda\phi(x)|)^{(1-\beta)/\alpha} exp\left(\Re\left((\zeta\lambda\phi(x))^{1/\alpha}\right)\right) + \frac{C_2}{1+|\zeta\lambda\phi(x)|}, \quad (1.6)$$

where, $\zeta \in C$, $|arg(\zeta\lambda\phi(x))| \le \mu$.

We are interested in particular in the behavior of $I_{\alpha,\beta}(\lambda)$ when λ is large, as for small λ the integral is just bounded.

Results and discussions

In this section we consider integral (1.2) in the finite interval $[a, b] \subset R$, $-\infty < a < b < +\infty$, as well as we are interested in a particular case of the integral (1.2), when $0 < \alpha < 2, \beta = \alpha$, i.e.

$$I_{\alpha,\alpha}(\lambda) = \int_I E_{\alpha,\alpha}(\zeta\lambda\phi(x))\psi(x)dx. \quad (2.1)$$

Since $I_{\alpha,\alpha}(\lambda)$ is bounded for small λ, further we can assume that $\lambda \ge 1$.

Theorem 2.1. Let $-\infty < a < b < +\infty$ and $I = [a, b] \subset R$. Let $0 < \alpha < 2$ and let ϕ be a real-valued function such that $\phi \in C^k(I), k \ge 1$. Let $\phi^k(x) \vee\ge 1$ for all $x \in I$, and $\zeta \in C, \vee \zeta \vee \ne 0$, then

$$\left|\int_I E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx\right| \le M_k\lambda^{\frac{-1}{k}}, \lambda \ge 1 \quad (2.2)$$

for k = 1 and ϕ' is monotonic, or k ≥ 2. Here M_k does not depend on λ.

We note that the classical van der Corput lemma (1.1) is covered by (2.2) with α = 1.

Proof. First we will prove the case k = 1. Let $0 < \alpha < 2$, $\lambda \ge 1$ and let ϕ has one zero $c \in [a, b]$. Let us consider the integral

$$\int_I E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx.$$

Then integrating by parts gives

$$\int_I E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx = \frac{\alpha}{\zeta\lambda}\int_I \frac{d}{d\phi(x)}\Big(E_{\alpha,1}(\zeta\phi(x))\Big)\,dx$$

$$\frac{\alpha}{\zeta\lambda}\int_I \frac{1}{\phi'(x)}\frac{d}{dx}\Big(E_{\alpha,1}(\zeta\lambda\phi(x))\Big)\,dx$$

$$\frac{\alpha}{\zeta\lambda}E_{\alpha,1}(\zeta\lambda\phi(b))\frac{1}{\phi'(b)} - \frac{\alpha}{\zeta\lambda}E_{\alpha,1}(\zeta\lambda\phi(a))\frac{1}{\phi'(a)}$$

$$\frac{-\alpha}{\zeta\lambda}\int_I E_{\alpha,1}(\zeta\lambda\phi(x))\frac{d}{dx}\left(\frac{1}{\phi'(x)}\right)dx,$$

thanks to property $\frac{d}{dz}E_{\alpha,1}(z) = \frac{1}{\alpha}E_{\alpha,\alpha}(z)$ and Then we get

$$\left|\int_I E_{\alpha,\alpha}(\zeta\lambda\phi(x))\,dx\right| \le \frac{\alpha}{\lambda}\int_I |E_{\alpha,1}(\zeta\lambda\phi(x))|\left|\frac{d}{dx}\left(\frac{1}{\phi'(x)}\right)\right|dx$$

$$\frac{+\alpha}{\lambda}|E_{\alpha,1}(\zeta\lambda\phi(x))|\frac{1}{|\phi'(b)|}$$

$$\frac{+\alpha}{\lambda}|E_{\alpha,1}(\zeta\lambda\phi(a))|\frac{1}{|\phi'(a)|}. \tag{2.3}$$

As ф’ is monotonic and $\phi'(x) \ge 1$ for all $x \in [a,b]$, then $\frac{1}{\phi'}$ is also monotonic, and $\frac{d}{dx}\frac{1}{\phi'(x)}$ has a fixed sign.

Hence estimate (1.6) and $\phi(c) = 0, c \in [a,b]$ gives

$$\left|\int_I E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx\right| \le \frac{C\alpha}{\lambda}\int_I (1+\lambda|\phi(x)|)^{\frac{1-\alpha}{\alpha}}$$

$$\frac{+C\alpha}{\lambda}(1+\lambda|\phi(b)|)^{\frac{1-\alpha}{\alpha}} + \frac{C\alpha}{\lambda}(1+\lambda|\phi(a)|)^{\frac{1-\alpha}{\alpha}}$$

$$\le^{|\phi(x)|\ge 0}\frac{C\alpha}{\lambda}\int_I \left|\frac{d}{dx}\left(\frac{1}{\phi'(x)}\right)\right|dx + \frac{2C\alpha}{\lambda}$$

$$\le \frac{C\alpha}{\lambda}\int_I \left|\frac{d}{dx}\left(\frac{1}{\phi'(x)}\right)\right|dx + \frac{2C\alpha}{\lambda}$$

$$\le \frac{C\alpha}{\lambda}\left[2 + \frac{1}{|\phi'(b)|} + \frac{1}{|\phi'(b)|}\right] \le \frac{M_1}{\lambda},$$

thanks to fixed sign $\frac{d}{dx}\frac{1}{\phi'(x)}$. Here M_1 does not depend on

We prove (2.1) for k = 2. Let for $d \in [a,b]$ satisfies $|\phi'(d)| \le |\phi'(x)|$ for all $x \in [a,b]$.

Then $|\phi'(x)| \ge \varepsilon$ on $[a,b]d-\varepsilon, d+\varepsilon$. Hence

$$\int_I E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx = \left(\int_\alpha^{d-\varepsilon} + \int_{d-\varepsilon}^{d+\varepsilon} + \int_{d+\varepsilon}^{b}\right)E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx.$$

Then $|\phi'(x)| \ge \varepsilon$ on $[a,b]d-\varepsilon, d+\varepsilon$, we obtain estimates

$$\left|\int_\alpha^{d-\varepsilon} E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx\right| \le M_1(\varepsilon\lambda)^{-1},$$

$$\left|\int_{d+\varepsilon}^{b} E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx\right| \le M_1(\varepsilon\lambda)^{-1}.$$

As

$$\left|\int_{d-\varepsilon}^{d+\varepsilon} E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx\right| \le 2\varepsilon,$$

it is evident that

$$\left|\int_\alpha^b E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx\right| \le 2M_1(\varepsilon\lambda)^{-1} + 2\varepsilon.$$

Taking $\varepsilon = \lambda^{\frac{-1}{2}}$ we obtain the estimate (2.1) for k=2.

We prove the case k ≥ 2 by induction method. Let (2.1) is true for k, and suppose $|\phi^{(k+1)}(x)| \geq 1$, for all $x \in [a,b]$, we prove (2.1) for k + 1.

Let for $d \in [a,b]$ satisfies $|\phi^k(d)| \leq |\phi^k(x)|$ for all $x \in [a,b]$.

Then on $|\phi^k(x)| \geq \varepsilon$ on $[a,b]c-\varepsilon, c+\varepsilon$. Therefore

As
$$\int_I E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx =$$

By inductive hypothesis, we infer that

$$\left|\int_\alpha^{d-\varepsilon} E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx\right| \leq M_k(\varepsilon\lambda)^{\frac{-1}{k}},$$

and

$$\left|\int_\alpha^{d-\varepsilon} E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx\right| \leq M_k(\varepsilon\lambda)^{\frac{-1}{k}},$$

As
$$\left|\int_{d-\varepsilon}^{d+\varepsilon} E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx\right| \leq 2\varepsilon,$$

we have

$$\left|\int_\alpha^b E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx\right| \leq 2M_k(\varepsilon\lambda)^{\frac{-1}{k}} + 2\varepsilon.$$

Taking $\varepsilon = \lambda^{\frac{-1}{k+1}}$ we obtain the estimate (2.1) for k+1.

Below we show that if ф' is not monotonic, then to obtain estimate (2.1) when k = 1, it is necessary to increase the smoothness of function ф.

Theorem 2.2. Let $-\infty < a < b < +\infty$ and $I = [a;b] \subset R$. Let $0 < \alpha < 2$ and let ф be a real-valued function such that $\phi \in C^2(I)$. Let $|\phi'(x)| \geq 1$ for all $x \in I$, and $\zeta \in C$, then

$$\left|\int_I E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx\right| \leq M\lambda^{-1}, \lambda \geq 1,$$

where M does not depend on λ.

Proof. Suppose that $\phi \in C^2(I)$ and $|\phi'(x)| \geq 1$ for all $x \in I$, then from (2.3) we have

$$\left|\int_I E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx\right| \leq \frac{\alpha}{\lambda}\int_I |E_{\alpha,1}(\zeta\lambda\phi(x))|\left|\frac{d}{dx}\left(\frac{1}{\phi'(x)}\right)\right|dx$$
$$+\frac{\alpha}{\lambda}|E_{\alpha,1}(\zeta\lambda\phi(b))|$$
$$+\frac{\alpha}{\lambda}|E_{\alpha,1}(\zeta\lambda\phi(a))|.$$

Since $\phi \in C^2(I)$ and $|\phi'(x)| \geq 1$ for all $x \in I$,, then the function will be continuous and bounded, and therefore by (1.6) we have

$$\left|\int_I E_{\alpha,\alpha}(\zeta\lambda\phi(x))dx\right| \leq \frac{C\alpha}{\lambda}\left[\int_I \left|\frac{d}{dx}\left(\frac{1}{\phi'(x)}\right)\right|dx + 2\right]$$
$$\leq \frac{C\alpha}{\lambda},$$

where $M_1 = \left\|\frac{d}{dx}\left(\frac{1}{\phi'(x)}\right)\right\|_{L^\infty(I)}, C \geq |E_{\alpha,1}(z)|,$ and M is a constant independent of λ.

Conclusion

We could show the behavior of $I_{\alpha,\beta}(\lambda)$ when λ is large, and we obtain variety of van der Corput type lemmas for the integral (2.1).

Acknowledgement

This work was supported by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan, project no. AP09259578.

References

1 Ruzhansky M. and Torebek B.T. (2021) Van der Corput lemmas for Mittag-Leffler functions, II, α-directions, Bulletin des Sciences Mathermatiques, 171:103016, pp. 1–23.

2 Xiao L. (2017) Endpoint estimates for one-dimensional oscillatory integral operators, Adv. Math., 316, pp. 255–291.

3 Stein E.M. (1993) Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals, Princeton Mathematical Series., vol. 43., Princeton Univ. Press, Princeton.

4 Greenblatt M. (2005) Sharp estimates for one-dimensional oscillatory integral operators with phase, Amer. J. Math., 127:3, pp. 659–695.

5 Stein E.M., Wainger S. (1970) The estimation of an integral arising in multiplier transformations, Studia Math., 35, pp.101–104.

6 Phong D.H., Stein E.M. (1994) Operator versions of the van der Corput lemma and Fourier integral operators, Math. Res. Lett.,1, pp. 27–33.

7 Phong D.H., Stein E.M. (1994) Models of degenerate Fourier integral operators and Radon transforms, Ann. Math.,140:3, pp.703–722.

8 Rogers K.M. (2005) Sharp van der Corput estimates and minimal divided differences, Proc. Amer. Math. Soc., 133: 12, pp. 3543–3550.

9 Parissis I.R. (2008) A sharp bound for the Stein-Wainger oscillatory integral, Proc. Amer. Math. Soc., 136:3, pp. 963–972.

10 Xiao L. (2017) Endpoint estimates for one-dimensional oscillatory integral operators, Adv. Math., 316, pp. 255–291.

11 Bourgain J., Guth L. (2011) Bounds on oscillatory integral operators based on multilinear estimates, Geom. Funct. Anal., 21:6, pp. 1239–1295.

12 Carbery A., Christ M. and Wright J. (1999) Multidimensional van der Corput and sublevel set estimates, J. Amer. Math. Soc., pp.981–1015.

13 Christ M., Li X., Tao T., Thiele C. (2005) On multilinear oscillatory integrals, nonsingular and singular, Duke Math. J., 130:2, pp. 321–351.

14 Greenleaf A., Pramanik M. and Tang W. (2007) Oscillatory integral operators with homoge-neous polynomial phases in several variables, J. Funct. Anal., 244:2, pp. 444–487.

15 Phong D.H., Stein E.M. and Sturm J. (2001) Multilinear level set operators, oscillatory integral operators, and Newton polyhedra, Math. Ann., 319, pp. 573–596.

16 Kamotski I. and Ruzhansky M. (2007) Regularity properties, representation of solutions and spectral asymptotics of systems with multiplicities, Comm. Partial Dierential Equations, 32, pp. 1–35.

17 Ruzhansky M. (2012) Multidimensional decay in the van der Corput lemma, Studia Math., 208, pp.1–10.

18 Kilbas A.A., Srivastava H.M., Trujillo J.J. (2006) Theory and Applications of Fractional Dierential Equations, North-Holland Mathematics Studies.

19 Goreno R., Kilbas A.A., Mainardi F., Rogosin S.V. (2014) Mittag-Leer Functions, Related Topics and Applications, Springer Monographs in Mathematics, Springer, Heidelberg.

Information about author

Beisenbay Aray Amanzholkyzy

Master, PhD student of al-Farabi Kazakh National University, Junior researcher of the Institute of Mathematics and Mathematical Modeling,125 Pushkin str., 050010, Almaty, Kazakhstan

ORCID ID: 0000-0001-5032-2428

E-mail: beisenbay@math.kz

Автор туралы мәлімет

Бейсенбай Арай Аманжолқызы

Магистр, әл-Фараби атындағы ҚазҰУ докторанты, Математика және математикалық модельдеу институтының кіші ғылыми қызметкері, 050010, Алматы қ., Пушкин көш., 125, Қазақстан

ORCID ID: 0000-0001-5032-2428
E-mail: beisenbay@math.kz

Информация об авторе

Бейсенбай Арай Аманжолкызы
Магистр, докторант КазНУ им. аль-Фараби, младший научный сотрудник Института математики и математического моделирования, 050010, г. Алматы, ул. Пушкина, 125, Казахстан
ORCID ID: 0000-0001-5032-2428
E-mail: beisenbay@math.kz

Бейсенбай А.А.

Математика және математикалық модельдеу институты, 050010, Алматы қ., Қазақстан
әл-Фараби атындағы ҚазҰУ, 050040, Алматы қ., Қазақстан
E-mail: beisenbay@math.kz

ВАН ДЕР КОРПУТ ЛЕММАЛАРЫНЫҢ АНАЛОГТАРЫ ТУРАЛЫ

Аңдатпа. Бұл мақалада Миттаг-Леффлер функцияларымен Ван дер Корпут леммаларының аналогтары қарастырылады. Шын мәнінде, бұл бағалау ең алғаш рет голланд математигі Йоханнес Хольтерус ван дер Корпутпен алынды және оның құрметіне Ван дер Корпут леммасы деп аталды. Ван дер Корпут леммаларының түрлі жалпыламалары бірнеше жыл бойы зерттеліп келеді. Жалпы мақсатымыз бен жұмыс барысын сипаттайтын болсақ: біз Ван дер Корпут леммасындағы экспоненциалды функцияны Миттаг-Леффлер типті функциямен ауыстырамыз, яғни уақыттан тәуелсіз дифференциалдық теңдеулерді талдау кезінде пайда болатын тербелмелі интегралдарды зерттейміз. Арнайы осы мақсат үшін бұл мақалада диапазоны $0 < \alpha < 2$ болатын

$$I_{\alpha,\beta}(\lambda) = \int_R E_{\alpha,\beta}\big(\zeta\lambda\phi(x)\big)\psi(x)dx,$$

интегралдың $\beta = \alpha$ жағдайына назар аударылады. Бұл $E_{\alpha,\beta}\big(i^{\alpha}\lambda\phi(x)\big)$ функциялары қатысқан тербелмелі интегралдар [1] жұмысында зерттеліп алынған бағалаулардың әртүрлілігін кеңейтеді. Бұл мақалада бізді λ шамасы үлкен болған жайғдайдағы $I_{\alpha,\beta}(\lambda)$ тербелмелі интегралдың өзгеру әрекеті қызықтырады. Біз қарастырған тербелмелі интегралдың кему жылдамдықтары β және α параметрлерінің диапазондарына байланысты екенін түсінеміз. Ван дер Корпут леммаларын одан әрі тереңнен зерттеу математикада және одан өзге салаларда жаңа түсініктер мен қолданбалардың ашылуына мүмкіндік береді.

Тірек сөздер: Ван дер Корпут леммалары, Миттаг-Леффлер функциясы, индуктивті гипотеза, бағалау, тегіс функция, тербелмелі интегралдар.

Бейсенбай А.А
Институт математики и математического моделирования, 050010, г. Алматы, Казахстан
КазНУ им. Аль-Фараби, 050040, г. Алматы, Казахстан
E-mail: beisenbay@math.kz

ОБ АНАЛОГАХ ЛЕММ ВАН ДЕР КОРПУТА

Аннотация. В данной статье рассматриваются аналоги лемм Ван дер Корпута с функциями Миттаг-Леффлера. На самом деле эта оценка была впервые получена голландским математиком Йоханнесом Хольтерусом ван дер Корпутом и названа в его честь леммой Ван дер Корпута. На протяжении многих лет изучались различные

обобщения лемм Ван дер Корпута. Если описать нашу общую цель и ход работы: мы заменяем экспоненциальную функцию в лемме ван дер Корпута функцией типа Миттаг-Леффлера, то есть изучаем осциллирующие интегралы, возникающие при анализе дифференциальных уравнений, не зависящих от времени. Специально для этой цели в настоящей статье основное внимание уделяется интегралу формы

$$I_{\alpha,\beta}(\lambda) = \int_R E_{\alpha,\beta}\big(\zeta\lambda\phi(x)\big)\psi(x)dx,$$

для диапазона 0 < α < 2, в случае β = α. Это расширяет разнообразие оценок, полученных в работе [1], где изучались интегралы с функциями $E_{\alpha,\beta}\big(i^{\alpha}\lambda\phi(x)\big)$. В этой статье нас интересует поведение интеграла $I_{\alpha,\beta}(\lambda)$ при больших λ. Мы понимаем, что скорости убывания рассмотренного нами интеграла зависят от диапазонов параметров α и β. Дальнейшее изучение лемм Ван дер Корпута откроет новые идеи и приложения в математике и за ее пределами.

Ключевые слова: леммы Ван дер Корпута, функция Миттаг-Леффлера, гипотеза индукции, оценка, гладкая функция, осциллирующие интегралы.