UDC 510.67
IRSTI 27.03.66
https://doi.org/10.55452/1998-6688-2023-20-2-67-72

Markhabatov N.D.
Eurasian National University. L.N. Gumilyov, 010008, Astana, Kazakhstan
email: nur_24.08.93@mail.ru

APPROXIMATIONS OF THE THEORIES OF STRUCTURES WITH ONE EQUIVALENCE RELATION

Abstract. Recently, various methods similar to the "transfer principle" have been rapidly developing, where one property of a structure or pieces of this structure is satisfied in all infinite structures or in another algebraic structure. Such methods include smoothly approximable structures, holographic structures, almost sure theories, and pseudofinite structures approximable by finite structures. Pseudofinite structures are mathematical structures that resemble finite structures but are not actually finite. They are important in various areas of mathematics, including model theory and algebraic geometry. Pseudofinite structures are a fascinating area of mathematical logic that bridge the gap between finite and infinite structures. They allow studying infinite structures in ways that resemble finite structures, and they provide a connection to various other concepts in model theory. Further studying pseudofinite structures will continue to reveal new insights and applications in mathematics and beyond. Pseudofinite theory is a branch of mathematical logic that studies structures that are similar in some ways to finite structures, but can be infinitely large in other ways. It is an area of research that lies at the intersection of model theory and number theory and deals with infinite structures that share some properties with finite structures, such as having only finitely many elements up to isomorphism. A. Lachlan introduced the concept of smoothly approximable structures in order to change the direction of analysis from finite to infinite, that is, to classify large finite structures that seem to be smooth approximations to an infinite limit. The theory of pseudofinite structures is particularly relevant for studying equivalence relations. In this paper, we study the model-theoretic property of the theory of equivalence relations, in particular, the property of smooth approximability. Let L = {E}, where E is an equivalence relation. We prove that an any ω-categorical L-structure M is smoothly approximable. We also prove that any infinite L-structure M is pseudofinite.

Key words: pseudofinite structure, pseudofinite theory, equivalence relation, approximation, approximation of theory, smoothly approximable structure.

Мархабатов Н.Д.
Л.Н.Гумилев атындағы Еуразия Ұлттық университеті, 010008, Астана қ., Қазақстан
email: nur_24.08.93@mail.ru

БІР ЭКВИВАЛЕНТТІК ҚАТЫНАСЫ БАР ҚҰРЫЛЫМ ТЕОРИЯЛАРЫНЫҢ АППРОКСИМАЦИЯЛАРЫ

Аңдатпа. Соңғы уақытта құрылымның бір қасиеті немесе осы құрылымның бөліктері барлық шексіз құрылымдарда немесе басқа алгебралық құрылымда қанағаттандырылатындай «тасымалдау принципіне» ұқсас әртүрлі әдістер қарқынды дамып келеді. Мұндай әдістерге біркелкі жуықталатын құрылымдар, голографиялық құрылымдар, сенімді дерлік теориялар және ақырлы құрылымдармен жуықталатын псевдоақырлы құрылымдар жатады. Псевдоақырлы құрылымдар - ақырлы құрылымдарға ұқсайтын, бірақ шын мәнінде ақырлы емес математикалық құрылымдар. Олар математиканың әртүрлі салаларында, соның ішінде модельдер теориясы мен алгебралық геометрияда маңызды. Псевдоақырлы құрылымдар - бұл ақырлы және шексіз құрылымдар арасындағы алшақтықты өтейтін математикалық логиканың қызықты саласы. Олар шексіз құрылымдарды ақырлы құрылымдарды еске түсіретін тәсілдермен зерттеуге мүмкіндік береді және әртүрлі басқа теориялық модельдік тұжырымдамаларына сілтемелер береді. Псевдоақырлы құрылымдарды одан әрі зерттеу математикада және одан тыс жерлерде жаңа идеялар мен қолданбаларды ашуды жалғастырады. Псевдоақырлы теориялар – кейбір жағынан шекті құрылымдарға ұқсас, бірақ басқа аспектілері бойынша шексіз үлкен болуы мүмкін құрылымдарды зерттейтін математикалық логиканың бөлімі. Бұл модельдер теориясы мен сандар теориясының қиылысында орналасқан және изоморфизмге дейінгі элементтердің шектеулі саны сияқты шекті құрылымдармен ортақ қасиеттері бар шексіз құрылымдармен айналысатын зерттеу саласы. А. Лахлан талдаудың бағытын ақырлыдан шексізге өзгерту, яғни шексіз шекке тегіс аппроксимациялау болып көрінетін үлкен шекті құрылымдарды жіктеу үшін құрылымдардың біркелкі аппроксимациялау түсінігін енгізді. Псевдоақырлы құрылымдар теориясы эквиваленттік қатынастарды зерттеу үшін ерекше өзекті болып табылады. Бұл жұмыста біз эквиваленттік қатынастар теориясының модельдік-теориялық қасиетін, атап айтқанда, тегіс аппроксимациалану қасиетін зерттейміз. Егер L = {E} және *E* – *L*-құрылымда эквиваленттік қатынас болса, онда кез келген ω-категориялық *L*-құрылым *M* тегіс аппроксимациаланатыны дәлелденді. Сонымен қатар, кез келген шексіз *L*-құрылым *M* псевдоақырлы болып табылады.

Тірек сөздер: псевдоақырлы құрылым, псевдоақырлы теория, эквиваленттік қатынас, аппроксимация, теория аппроксимациясы, тегіс аппроксимацияланатын құрылым.

Мархабатов Н.Д.
Евразийский Национальный университет им. Л.Н.Гумилева,, 010008, г. Астана, Казахстан
email: nur_24.08.93@mail.ru

АППРОКСИМАЦИИ ТЕОРИЙ СТРУКТУР С ОДНИМ ОТНОШЕНИЕМ ЭКВИВАЛЕНТНОСТИ

Аннотация: В последнее время бурно развиваются различные методы, схожие к «принципу переноса», когда одно свойство структуры или частей этой структуры выполняется во всех бесконечных структурах или в другой алгебраической структуре. К таким методам относятся гладко аппроксимируемые структуры, голографические структуры, почти надежные теории и псевдоконечных структуры, аппроксимируемые конечными структурами. Псевдоконечные структуры — это математические структуры, которые напоминают конечные структуры, но на самом деле не являются конечными. Они важны в различных областях математики, включая теорию моделей и алгебраическую геометрию. Псевдоконечные структуры — это увлекательная область математической логики, которая устраняет разрыв между конечными и бесконечными структурами. Они позволяют изучать бесконечные структуры способами, напоминающими конечные структуры, и обеспечивают связь с различными другими концепциями теории моделей. Дальнейшее изучение псевдоконечных структур будет продолжать открывать новые идеи и приложения в математике и за ее пределами. Псевдоконечные теории — это раздел математической логики, изучающий структуры, которые в чем-то похожи на конечные структуры, но могут быть бесконечно большими в других отношениях. Это область исследований, которая находится на пересечении теории моделей и теории чисел и имеет дело с бесконечными структурами, которые имеют некоторые общие свойства с конечными структурами, например, имеют только конечное число элементов с точностью до изоморфизма. А. Лахлан ввел понятие гладко аппроксимируемых структур, чтобы изменить направление анализа с конечного на бесконечное, т. е. классифицировать большие конечные структуры, которые кажутся гладкими приближениями к бесконечному пределу. Теория псевдоконечных структур особенно актуальна для изучения отношений эквивалентности. В данной работе исследуется теоретико-модельное свойство теории отношений эквивалентности, в частности, свойство псевдоконечности. Пусть L = {E}, где *E* – отношение эквивалентности на *L*-структуре. Доказано, что любая ω-категоричная *L*-структура *M* гладко аппроксимируема. Также доказано, что любая бесконечная *L*-структура *M* является псевдоконечной.

Ключевые слова: псевдоконечная структура, псевдоконечная теория, отношение эквивалентности, аппроксимация, аппроксимация теории, гладко аппроксимируемая структура.

Introduction

Approximations for permutation theory by theories of finite structures were studied in [1], for locally free algebras in [2], for ordered theories in [3], for Abelian groups in terms of Szmielev invariants in [4,5], for regular graphs studied in [6].

We denote by $\mathcal{T}_L$ the family of all theories of the language *L*. It should be noted that in [14] the rank for families of theories was introduced similarly to the Morley rank. And in [13], the following proposition was proved

Proposition 1. *If L is a language containing an m-ary predicate symbol, for m ≥ 2, or an n-ary functional symbol, for n ≥ 1, then* $RS(\mathcal{T}_L) = \infty$.

In paper [7] isomorphisms and algorithmic properties of structures with two equivalences are considered based on methods developed by D.A. Tusupov for graph definability in a bipartite graph and in a structure with two equivalences, which preserve the algorithmic and syntactic properties of the original structure.

Definition 1. [8] An infinite structure *M* is *pseudofinite* if for any sentence φ with M |= φ, there exists a finite model M_0 with M_0 |= φ.

Definition 2. Let *M* and *N* be *L*-structures. *N* is a *homogeneous substructure* of *M*, notationally $N \leq_{hom} M$, if $N \leq M$ and for every positive natural k and every pair $\bar{a}, \bar{b} \in N^k$, $\bar{a}$ and *b* lie in the same $Aut(M)$-orbit if and only if $\bar{a}$ and $\bar{b}$ lie in the same $Aut_{\{N\}}(M)$-orbit, where $Aut_{\{N\}}(M) := \{\sigma \in Aut(M) : \sigma(N) = N\}$.

Definition 3. An *ω*-categorical structure *ω* is called *smoothly approximable* if it is the union of an ω-chain of finite homogeneous substructures; or, that is the same, if any proposition $\varphi \in Th(M)$ is true for some finite homogeneous substructure N of the structure M.

Smoothly approximable structures were first considered in general form in [9], then in [10]. The theory of models of smoothly approximable structures was further developed in the works by G. Cherlin and E. Hrushovskii [11,12]. The class of smoothly approximable structures is the class of omega-categorical supersimple structures of finite rank, which properly contains the class of omega-categorical omega-stable structures (in particular, totally categorical structures).

Remark 1. [15] *Any smoothly approximated structure is pseudofinite, the converse is not always true.*

Example 1. *Random graph is ω-categorical, homogeneous, pseudofinite, but not smoothly approximable.*

Main provisions

We are dealing with (complete) theories of equivalence relations. We fix a language L = {E} and a L-structure M, where E is interpreted in *M* as an equivalence relation.

Let the set $\{\kappa_i | i \in \omega\}$ be the set of cardinalities of *E*-classes of the *L*-structure *M*. To describe the smooth approximability of *L*-structure *M*, we use the following invariants. Consider the pair $(\kappa_i, \lambda(\kappa_i))$ where k_i is the cardinality of the *i*-th *E*-class, and $\lambda(\kappa_i)$ is the number of cardinality classes of k_i, $\lambda(\kappa_i) \in \omega \cup \{\infty\}$. Therefore, we also consider the indicator $\varepsilon \in \{0,1\}$ indicating the absence or presence of an infinite *E*-class. Also, we often use the following well-known

Fact: An *L*-theory *T is ω-categorical iff the set* $\{\kappa_i | i \in \omega\}$ *is finite and all models have the same invariants* $(\kappa_i, \lambda(\kappa_i))$.

If *M* is countably categorical, there may be classes of infinite cardinality among the *E*-classes, that is, in the set $\{\kappa_i | i \in \omega\}$ there is at least one infinite cardinality k_i.

Proposition 2. *Any an ω-categorical L-structure M is smoothly approximable.*

The following theorem says that any *L*-structure *M* is approximated by finite structures.

Theorem 3. *Any infinite* L*-structure M is pseudofinite.*

Materials and Methods

We know several ways to show that a theory *T* is pseudofinite [8], that is, approximated by theories of finite structures.

We have the following methods (tools) for examining algebraic structures for pseudofiniteness:

— *Probabilistic argument [15,16];*

— *Ultraproduct Construction [8,17];*

— *Approximations of the theory of algebraic structures by theories of finite structures[18];*

— *Fraisse limit.*

— *Smoothly Approximability [9]*

Some of the above model-theoretic methods are also actively used for applied research. For example, the *Probabilistic method* is widely used in data analysis and can provide valuable information about relationships between variables in a dataset. However, it is important to select the appropriate method for a particular research question and carefully evaluate the assumptions underlying the model.

Results and Discussion

Proof of the **Proposition 2:** Let *M* be countably categorical and $\varepsilon = 0$. Then either all *E*-classes have the same cardinality i.e. for any *E*-class there is an infinite number of isomorphic *E*-classes (1), or some finite number of *E*-classes have the same cardinality (2). Since all elements of each *E*-class are equivalent, there is an automorphism mapping this class onto itself. This means that each equivalence class can be considered as a finitely homogeneous substructure. Therefore, (1) if for all i and j, $\kappa_i = \kappa_j$, then M is an unique up to isomorphism and $M = \bigcup_{i\in\omega} M_i$, where $M_i = \coprod_{i\in\omega} N_i$ and N_i are *E*-classes.

(2) Let the set of cardinalities of *E*-classes be finite in *M*, that is, $\kappa_0, \kappa_1, \dots, \kappa_{n-1}$. For finite natural l and m, with $1 \le l \le m$ and $0 < p < n-1$ denote by A a finite substructure consisting of $\lambda(\kappa_p) = l$ *E*-classes. This substructure is finitely homogeneous and can be represented as a disjoint union of a finite number of *E*-classes. Now we denote by B a substructure that for any $p < t \le n-1$, $\lambda(\kappa_t) = \infty$. Then $B = \coprod_{i\in\omega}(N_i^1 \sqcup N_i^2 \sqcup \dots \sqcup N_i^t)$. Generally, $M = \bigcup_{i\in\omega} M_i$, where $M_i = A \sqcup B$.

Let M be countably categorical and $\varepsilon = 1$. This means that among $\kappa_0, \kappa_1, \dots, \kappa_{n-1}$ only one indicates infinite cardinality. Let's say it k_0. And all other $\{\kappa_i | 1 \le i \le n-1\}$ indicates finite cardinalities. Let us denote the substructure with finite cardinalities as in the previous case by A. All possible cases for A are considered in the previous part of the proof.

And the substructure consisting of infinite classes will be denoted by B. If $\lambda(\kappa_0)$ is finite, let's say $\lambda(\kappa_0) = s$, for finite natural s, then $B = \bigcup_{i\in\omega}(N_i^1 \sqcup N_i^2 \sqcup \dots \sqcup N_i^s)$ at that $N_j^s \subset N_i^s$, for $j < i$. If $\lambda(\kappa_0)$ is infinite, then $B = \coprod_{i\in\omega} C_i$, where $C_i = \bigcup_{i\in\omega}(N_i^1 \sqcup N_i^2 \sqcup \dots \sqcup N_i^s)$.

Generally, $M = \bigcup_{i\in\omega} M_i$, where $M_i = A \sqcup B$.

Proof of the **Theorem 3:** The proof of pseudofiniteness is in some sense easier than smooth approximability, because smooth approximability is a stronger condition including countably categorical and finitely homogeneity substructures. Therefore, we consider only the non-countably categorical case, if the structure M is countably categorical, then we refer to the smooth approximability. Table 1 below shows the proof.

Table 1 – All possible approximations by finite structures of *L*-structure *M*.

		The number of equivalence classes	
		$\lambda(\kappa_i)$ is finite	$\lambda(\kappa_i)$ is infinite
k_i is finite	$\kappa_i < n$	**not infinite**	1) If the cardinality set is finite, then M is smoothly approximable, hence pseudofinite. Let the set of cardinalities infinite. Then $M = \coprod_{i\in\omega} M_i$ and M_i are *E*-classes.
	$\kappa_i = \kappa_j, i \ne j$	**not infinite**	2) The structure M is smoothly approximable by the first case of the proof of Proposition 2.
k_i is infinite	Only some classes have infinite cardinality	*3) L*-structure *M* represented as $M = M_0 \sqcup M_1$, where M_0 is a substructure of finite cardinality *E*-classes, M_0 is a substructure of finite cardinality *E*-classe. All possible cases were considered earlier. If it is countably categorical then it is smoothly approximable by Proposition 2. If not, then case 1) of this table. For M_1 we refer to Proposition 2	4) Similarly, by Proposition 2 and case 1) of this table.
	All equivalence classes have infinite cardinality;	5) Smoothly approximable, hence pseudofinite	6) Smoothly approximable, hence pseudofinite

Example. Let M be a countably infinite set and $L = \{=\}$. List M as $\{a_i : i \in \omega\}$, and $M_i = \{a_0, \dots, a_i\}$. Then M is smoothly approximable, and $M = \bigcup_{i\in\omega} M_i$.

Conclusion

Pseudofinite structures are a fascinating area of mathematical logic that bridge the gap between finite and infinite structures. They allow for the study of infinite structures in ways that resemble finite structures, and they provide a connection to various other concepts in model theory. Further study of pseudofinite structures will continue to reveal new insights and applications in mathematics and beyond. The theory of pseudofinite structures is particularly relevant for studying equivalence relations. This article examines the model-theoretic property of the theory of equivalence relations, in particular, the property of smoothly approximability. We prove that an any ω-categorical L-structure M is smoothly approximable. We also prove that any infinite L-structure M is pseudofinite.

Information on funding

The study was partially supported by the Scientific Committee for Education and Science of the Republic of Kazakhstan (AP19674850, AP19677451).

References

1 Markhabatov N.D. (2019) Ranks for Families of Permutation Theories, The Bulletin of Irkutsk State University. Series Mathematics, vol. 28, pp.85–94. https://doi.org/10.26516/1997-7670.2019.28.85

2 Markhabatov N.D. (2019) Pseudofiniteness of locally free algebras, Algebra and Model Theory 12, Collection of papers, eds. A.G. Pinus, E.N. Poroshenko, S.V. Sudoplatov. Novosibirsk: NSTU, pp. 41–46.

3 Kulpeshov B.Sh., Sudoplatov S.V. (2019) Ranks and approximations for families of ordered theories, Algebra and Model Theory 12, Collection of papers, eds. A.G. Pinus, E.N. Poroshenko, S.V. Sudoplatov. Novosibirsk: NSTU, pp. 32–40.

4 Pavlyuk I.I. (2020) Approximations for Theories of Abelian Groups, eds. I.I. Pavlyuk, S.V. Sudoplatov, Mathematics and Statistics, vol. 8, no. 2, pp. 220–224. https://doi.org/10.13189/ms.2020.080218

5 Pavlyuk I.I., Sudoplatov S.V. (2019) Ranks for families of theories of abelian groups, The Bulletin of Irkutsk State University, Series Mathematics, vol., 28, pp. 95–112. https://doi.org/10.26516/1997-7670.2019.28.95

6 Markhabatov N.D., Sudoplatov S.V. (2022) Approximations Of Regular Graphs, Herald of the Kazakh-British Technical University, 19:1, pp. 44–49 https://doi.org/10.55452/1998-6688-2022-19-1-44-49

7 Tussupov D.A. (2016) Isomorphisms and Algorithmic Properties of Structures with Two Equivalences, *Algebra Logic* **55**, pp. 50–57. https://doi.org/10.1007/s10469-016-9375-8

8 Ax J. (1968) The Elementary Theory of Finite Fields, Annals of Mathematics, vol. 88, no. 2, Annals of Mathematics, pp. 239–71. https://doi.org/10.2307/1970573

9 Kantor W.M., Liebeck M.W. and Macpherson H.D. (1989) $\aleph_0$-categorical structures smoothly approximated by finite substructures, Proceedings of the London Mathematical Society, vol. 59, pp. 439–463. https://doi.org/10.1112/plms/s3-59.3.439

10 Macpherson D. (1997). Homogeneous and Smoothly Approximated Structures. In: Hart, B.T., Lachlan, A.H., Valeriote, M.A. (eds) Algebraic Model Theory. NATO ASI Series, vol 496. Springer, Dordrecht. https://doi.org/10.1007/978-94-015-8923-9_7 ,

11 Cherlin G., and E. Hrushovski (2003) Finite Structures with Few Types, vol. 152, Annals of Mathematics Studies, Princeton University Press, Princeton.

12 Hrushovski E. (1993) Finite Structures with Few Types. In: Sauer N.W., Woodrow R.E., Sands B. (eds) Finite and Infinite Combinatorics in Sets and Logic. NATO ASI Series (Series C: Mathematical and Physical Sciences), vol 411. Springer, Dordrecht. https://doi.org/10.1007/978-94-011-2080-7

13 Markhabatov N.D., Sudoplatov S.V. (2021) Ranks for families of all theories of given languages, Eurasian Math. J., 12:2, pp. 52–58. https://doi.org/10.32523/2077-9879-2021-12-2-52-58

14 Sudoplatov S.V. (2021) Ranks for families of theories and their spectra~/ S.~V.~Sudoplatov// Lobachevskii J. Math 42, pp. 2959–2968. https://doi.org/10.1134/S1995080221120313

15 Gaifman H. (1964) Concerning measures in first-order calculi, Israel J. Math. 2, pp. 1–18. https://doi.org/10.1007/BF02759729

16 Lynch J. (1980) Almost sure theories, Annals of Mathematical Logic, 18, pp. 91–135. https://doi.org/10.1016/0003-4843(80)90014-5

17 Ax J., Kochen S. (1965) Diophantine problems over local fields I. Amer. J. Math. 87, pp. 605–630. https://doi.org/10.2307/2373065

18 Sudoplatov S.V. (2020) Approximations of theories, Siberian Electronic Mathematical Reports, vol. 17, pp. 715–725. https://doi.org/10.33048/semi.2020.17.049

Information about author

Markhabatov Nurlan Darkhanuly
Department of Algebra and Geometry, L.N. Gumilyov Eurasian National University, st. Kazhymukan 13, Almaty district, 010000, Astana, Kazakhstan
ORCID ID: 0000-0002-5088-0208
E-mail: nur_24.08.93@mail.ru

Автор туралы мәлімет

Мархабатов Нұрлан Дарханұлы
Л.Н. Гумилев ат. Еуразия ұлттық университетінің алгебра және геометрия кафедрасы, Қажымұқан көш., 13, Алматы ауданы, 010000, Астана қ., Қазақстан
ORCID ID: 0000-0002-5088-0208
E-mail: nur_24.08.93@mail.ru

Информация об авторе

Мархабатов Нурлан Дарханулы
Кафедра алгебры и геометрии Евразийского Национального университета им. Л.Н. Гумилева, ул. Кажымукана, 13, Алматинский район, 010000, г. Астана, Казахстан
ORCID ID: 0000-0002-5088-0208
E-mail: nur_24.08.93@mail.ru