UDK 512.544
IRSTI 27.17.23
https://doi.org/10.55452/1998-6688-2023-20-2-57-66

Mardanov N.A.
Kazakh-British Technical University, 050000, Almaty, Kazakhstan
Email: mardanov1602@gmail.com

TRIPLE TORTKEN IDENTITIES

Abstract. We define a triple Tortken product in Novikov algebras. Using computer algebra calculations, we give a list of polynomial identities up to degree 5 satisfied by Tortken triple product in every Novikov algebra. It has applications in theoretical physics, specifically in the field of quantum field theory and topological field theory. A Novikov algebra is defined as a vector space equipped with a binary operation called the Novikov bracket. The Jacobi identity ensures that the Novikov bracket behaves analogously to the commutator in Lie algebras. However, unlike Lie algebras, Novikov algebras are non-associative due to the presence of the Jacobi identity rather than the associativity condition. Novikov algebras find applications in theoretical physics, particularly in the study of topological field theories and quantum field theories on noncommutative spaces. They provide a framework for describing and analyzing certain algebraic structures that arise in these areas of physics. It's worth noting that Novikov algebras are a specific type of non-associative algebra, and there are various other types of non-associative algebras studied in mathematics and physics, each with its own defining properties and applications.

Key words: Novikov algebras, triple product, polynomial identities, Wolfram Mathematica.

Марданов Н.А.
Қазақстан-Британ техникалық университеті, 050000, Алматы қ., Қазақстан
Email: mardanov1602@gmail.com

ТЕРНАРЛЫ ТӨРТКЕН СӘЙКЕСТЕРІ

Андатпа. Біз Новиков алгебрасында тернарлы Торткен көбейтіндісін анықтаймыз. Компьютерлік алгебра есептеулерін пайдалана отырып, біз әрбір Новиков алгебрасында Торткен тернарлы көбейтіндісін қанағаттандыратын 5-дәрежеге дейінгі көпмүшелік сәйкестіктердің тізімін береміз. Оның теориялық физикада, әсіресе кванттық өріс теориясы және топологиялық өріс теориясы саласында қолданбалары бар. Новиков алгебрасы Новиков жақшасы деп аталатын екілік операциямен жабдықталған векторлық кеңістік ретінде анықталады. Якоби сәйкестендіру Новиков жақшасының Ли алгебраларындағы коммутаторға ұқсас әрекет ететінін қамтамасыз етеді. Алайда, Ли алгебраларынан айырмашылығы, Новиков алгебралары ассоциативтілік шартынан гөрі Якоби сәйкестігінің болуына байланысты ассоциативті емес. Новиков алгебралары теориялық физикада, әсіресе топологиялық өріс теорияларын және коммутативті емес кеңістіктердегі кванттық өріс теорияларын зерттеуде қосымшаларды табады. Олар физиканың осы салаларында пайда болатын кейбір алгебралық құрылымдарды сипаттау және талдау үшін негіз береді. Айта кетейік, Новиков алгебралары ассоциативті емес алгебраның белгілі бір түрі болып табылады және математика мен физикада зерттелетін ассоциативтік емес алгебралардың басқа да әр түрлі түрлері бар, олардың әрқайсысының өзіндік анықтаушы қасиеттері мен қолданбалары бар.

Тірек сөздер: Новиков алгебрасы, тернарлы көбейтіндісі, көп мүшелік сәйкестіктері, Вольфрам Математика.

Марданов Н.А.
Казахстанско-Британский технический университет, 050000, г. Алматы, Казахстан
Email: mardanov1602@gmail.com

ТЕРНАРНЫЕ ТОЖДЕСТВА ТОРТКЕНА

Аннотация. Мы определяем тернарное произведение Торткена в алгебрах Новикова. Используя вычисления компьютерной алгебры, мы даем список полиномиальных тождеств до степени 5, которым удовлетворяет тернарное произведение Торткена в каждой алгебре Новикова. Он имеет приложения в теоретической физике, особенно в области квантовой теории поля и топологической теории поля. Алгебра Новикова определяется как векторное пространство, снабженное бинарной операцией, называемой скобкой Новикова. Тождество Якоби гарантирует, что скобка Новикова ведет себя аналогично коммутатору в алгебрах Ли. Однако в отличие от алгебр Ли алгебры Новикова неассоциативны из-за наличия тождества Якоби, а не условия ассоциативности. Алгебры Новикова находят применение в теоретической физике, в частности, при изучении топологических теорий поля и квантовых теорий поля на некоммутативных пространствах. Они обеспечивают основу для описания и анализа определенных алгебраических структур, возникающих в этих областях физики. Стоит отметить, что алгебры Новикова представляют собой особый тип неассоциативной алгебры и существуют различные другие типы неассоциативных алгебр, изучаемых в математике и физике, каждый со своими определяющими свойствами и приложениями.
Ключевые слова: алгебра Новикова, тройное произведение, полиномиальные тождества, Вольфрам математика.

Inroduction

Let K be a field of characteristic zero. An algebra over K with identities

$$a(bc) = b(ac) \tag{1}$$

$$(a,b,c) = (a,c,b) \tag{2}$$

is called Novikov algebra, where $(a,b,c) = (ab)c - a(bc)$ is the associator of elements a, b and c. The identity (1) is called left-commutative and the identity (2) is called right- symmetric. Novikov algebras first appeared in the paper by Gelfand and Dorfman [7] in the study of Hamiltonian operators and then in the paper by Balinskii and Novikov [1] in the study of the classification of linear Poisson brackets. A simple example of Novikov algebra is the following. Let $A = K[x]$ be a polynomial algebra with multiplication $a \circ b = \partial(a)b$, where $\partial = \frac{\partial}{\partial x}$ is partial derivation. Let Novikov be a variety of Novikov algebras. For Novikov algebra A we define $A^{(-)} = (A, [,])$ and $A^{(+)} = (A, \{,\})$, where $[a,b] = ab - ba$ is the commutator and $\{a,b\} = ab + ba$ is the anti-commutator on the space $A \in \mathcal{N}ovikov$. Every Novikov algebra is Lie-admissible, that is, $A^{(-)}$ satisfies Jacobi identity. In addition, the commutator algebra of any Novikov algebra satisfies the identity

$$\sum_{\sigma \in S_4} (-1)^{\sigma} [x_{\sigma(1)}, [x_{\sigma(2)}, [x_{\sigma(3)}, [x_{\sigma(4)}, x_5]]]] = 0 \tag{3}$$

Dzhumadil'daev studied [3] Novikov algebras under anti-commutator and proved that that every algebra in $A^{\{(+)\}}$ satisfies so-called Tortken-identity

$$\{\{a,b\},\{c,d\} - \{a,d\},\{c,b\}\} = \{(a,b,c),d\} - \{(a,d,c),b\} \tag{4}$$

where $(a,b,c) = \{a,\{b,c\}\} - \{\{a,b\},c\}$ is the associator of elements a, b and c. Additionaly, it was demonstrated that Tortken and commutative identities imply any identity with a degree of no more than four. In [3], some examples of algebras that satisfy Tortken identity are provided. The following is one of them. Consider the polynomial algebra $A = K[x]$ with multiplication $a \circ b = \partial(ab)$ and the partial derivation $\partial = \frac{\partial}{\partial x}$. Then, at that point $(A,\circ)$ fulfills Tortken identity.

Main provisions

Tortken algebra is a commutative algebra that satisfies the Tortken identity. In the setting of Novikov algebras, these are the analogs of Jordan algebras. Any left-Zinbiel algebra is a Tortken algebra, as shown in [3]. An identity is called special for the anti-commutator product in a free Novikov algebra if it holds in all special Tortken algebras but does not hold in some Tortken algebras. It was shown in [4], there is an identity of degree 5 satisfied by algebras of $\mathcal{N}ovikov^{(+)}$ in addition to commutativity and the Tortken identity so-called besken

$$\left\{\left\{\{\{a,a\},a\},b\right\},b\right\}+\left\{\left\{\{\{a,b\},b\},a\right\},a\right\}+2\left\{\left\{\{\{a,a\},b\},b\right\},a\right\}+2\left\{\left\{\{\{a,b\},a\},a\right\},b\right\}$$
$$-3\left\{\left\{\{\{a,a\},b\},a\right\},b\right\}-3\left\{\left\{\{\{a,b\},a\},b\right\},a\right\}=0. \quad (5)$$

Moreover, Dzhumadil'daev [4] proved that commutative, Tortken and besken identities imply every identity of degree no more than seven satisfied by anti-commutator product in a free Novikov algebra. It remains an open problem to determine whether there are further new identities in degrees more than 7.

Methods

In this paper, we study special identities of Tortken algebras in the sense of ternary products. Define a triple Tortken product $\{\ ,\ ,\ \}: A\times A\times A\to A$ on a Novikov algebra A as follows

$$\{a,b,c\}=\{\{a,b\},c\},$$

where $\{a,b\}=ab+ba$ for all $a,b,c\in A$. We prove that there is no identity of degree three satisfied by the triple Tortken product in every Novikov algebra. To calculate triple Tortken identities in degree 5, we write a code on the software program Wolfram Mathematics and using it we demonstrate a list of polynomial identities of degree five by the triple Tortken product in every Novikov algebra. Triple identities of algebras were considered for many well-known classes of algebras such as Jordan, Lie [8] and Zinbiel algebras [2]. In [2], M. Bremner by using computer algebra studied special identities in terms of Tortkara triple product $[a,b,c]=\left[[a,b],c\right]$ in a free Zinbiel algebra and discovered one identity in degree 5 and one identity in degree 7 which do not follow from the identities of lower degree. It remains an open problem to determine whether there are further new identities in degree 9.

Results and discussion

In this section, we define the relation of differential algebras with Novikov algebras and then with Tortken algebras. Let $k\{x_1,\dots,x_n\}$ be a differenital polynomial algebra on variables $x_1,\dots,x_n$. For $f,g\in k\{x_1,\dots,x_n\}$ if $f\circ g=f'g$, then $(k\{x_1,\dots,x_n\},\circ)$ is a Novikov algebra. Assume that $N<x_1,\dots,x_n>$ - subalgebra of $(k\{x_1,\dots,x_n\},\circ)$ generated by $x_1,\dots,x_n$. Dzhumadil'daev and Löfwall (2002) proved that $N<x_1,\dots,x_n>$ is a free Novikov algebra over a field of characteristic zero and the set of all differential monomials $x_1^{(i_1)}x_2^{(i_2)}\dots x_n^{(i_n)}$ with $i_1+\dots+i_n=n-1$ forms a basis of $N<x_1,\dots,x_n>$.

Theorem 2.1. (Dzhumadil'daev,[5]) The multilinear dimension of free Novikov algebra in n variables is equal to $\binom{2(n-1)}{n-1}$.

Theorem 2.2. There is no identity of degree three satisfied by triple Tortken product in Novikov algebras over a field of characteristic zero.

Proof. Since we consider algebras over a field of characteristic zero, we can assume that all identities are multilinear, for details see [9, Chapter 1]. According to Theorem 2.1 in degree 3 there are $\binom{2(3-1)}{3-1}=\binom{4}{2}=6$ multilinear Novikov basis monomials:

$$a''bc,ab''c,abc'',a'b'c,a'bc',ab'c'$$

In degree 3 there are the following multilinear spanning monomials in generators a, b, c:

$$e_1 = \{a,b,c\}, \qquad e_2 = \{a,c,b\}, \qquad e_3 = \{b,c,a\},$$

Any triple Tortken identity of degree 3 can be written as a linear combination of e_1, e_2, e_3, that is,

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = 0 \qquad (6)$$

For some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{K}$. Note that

$$\{a,b,c\} = \{\{a,b\},c\} = (ab)c + (ba)c + c(ab) + c(ba) = (a'b)'c + (ab')'c + a'bc' + ab'c'$$
$$= a''bc + ab''c + 2a'b'c + a'bc' + ab'c',$$

$$\{a,c,b\} = a''bc + abc'' + 2a'bc' + a'b'c + ab'c',$$

$$\{b,c,a\} = ab''c + abc'' + 2ab'c' + a'b'c + a'bc'.$$

From (6) we derive

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$$
$$= a''bc(\lambda_1 + \lambda_2) + ab''c(\lambda_1 + \lambda_3) + abc''(\lambda_2 + \lambda_3) + a'b'c(2\lambda_1 + \lambda_2 + \lambda_3)$$
$$+ a'bc'(\lambda_1 + 2\lambda_2 + \lambda_3) + ab'c'(\lambda_1 + \lambda_2 + 2\lambda_3) = 0.$$

It leads to a system of linear equations

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ \lambda_1 + \lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 0 \\ 2\lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_1 + 2\lambda_2 + \lambda_3 = 0 \\ \lambda_1 + \lambda_2 + 2\lambda_3 = 0. \end{cases}$$

One can easily have that the system has only trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$. ■

Theorem 2.3. There exists a set of nontrivial polynomial identities of degree five by the triple Tortken product in every Novikov algebra over a field of characteristic zero.

Proof. According to Theorem 2.1 in degree 5 there are $\binom{2(5-1)}{5-1} = \binom{8}{4} = 70$ multilinear Novikov basis monomials

$$a''''bcde, ab''''cde, abc''''de, abcd''''e, abcde'''', a'''b'cde, a'''bc'de,$$

$$a'''bcd'e, a'''bcde', a''b''cde, a''bc''de, a''bcd''e, a''bcde'', a''b'c'de,$$

$$a''b'cd'e, a''b'cde', a''bc'd'e, a''bc'de', a''bcd'e', a'b''c'de, a'b''cd'e,$$

$$a'b''cde', ab''c'd'e, ab''c'de', ab''cd'e', a'b'c''de, a'bc''d'e, a'bc''de',$$

$$ab'c''d'e, ab'c''de', abc''d'e', a'b'cd''e, a'bc'd''e, a'bcd''e', ab'c'd''e,$$

$$ab'cd''e', abc'd''e', a'b'cde'', a'bc'de'', ab'c'de'', ab'cd'e'', abc'd'e''$$

$$ab''c''de, ab''cd''e, ab''cde'', abc''d''e, abc''de'', abcd''e'', a'b'''cde,$$

$$ab'''c'de, ab'''cd'e, ab'''cde', a'bc'''de, ab'c'''de, abc'''d'e, abc'''de',$$

$$a'bcd'''e, ab'cd'''e, abc'd'''e, abcd'''e', a'bcde''', ab'cde''', abc'de''',$$

$$abcd'e''', a'b'c'd'e, a'b'c'de', a'b'cd'e', a'bc'd'e', ab'c'd'e', a'bcd'e''.$$

In degree 5 there are 90 multilinear spanning monomials in generators a,b,c,d,e:
- 60 monomials $\{\{a^\sigma, b^\sigma, c^\sigma\}, d^\sigma, e^\sigma\}$ where $a^\sigma < b^\sigma$ in lex order, and
- 30 monomials $\{a^\sigma, b^\sigma, \{c^\sigma, d^\sigma, e^\sigma\}\}$ where $a^\sigma < b^\sigma$ and $c^\sigma < d^\sigma$ in lex order

For simplicity we write $\{a,b,c,d,e\}$ for $\{\{a,b,c\},d,e\}$ and [a,b,c,d,e] for {a,b,{c,d,e}}. Then we have

$$e_1 = \{a,b,c,d,e\}, e_2 = \{a,b,c,e,d\}, e_3 = \{a,b,d,c,e\}, e_4 = \{a,b,d,e,c\}, e_5 = \{a,b,e,c,d\},$$

$$e_6 = \{a,b,e,d,c\}, e_7 = \{a,c,b,d,e\}, e_8 = \{a,c,b,e,d\}, e_9 = \{a,c,d,b,e\}, e_{10} = \{a,c,d,e,b\},$$

$$e_{11} = \{a,c,e,b,d\}, e_{12} = \{a,c,e,d,b\}, e_{13} = \{a,d,b,c,e\}, e_{14} = \{a,d,b,e,c\}, e_{15} = \{a,d,c,b,e\},$$

$$e_{16} = \{a,d,c,e,b\}, e_{17} = \{a,d,e,b,c\}, e_{18} = \{a,d,e,c,b\}, e_{19} = \{a,e,b,c,d\}, e_{20} = \{a,e,b,d,c\},$$

$$e_{21} = \{a,e,c,b,d\}, e_{22} = \{a,e,c,d,b\}, e_{23} = \{a,e,d,b,c\}, e_{24} = \{a,e,d,c,b\}, e_{25} = \{b,c,a,d,e\},$$

$$e_{26} = \{b,c,a,e,d\}, e_{27} = \{b,c,d,a,e\}, e_{28} = \{b,c,d,e,a\}, e_{29} = \{b,c,e,a,d\}, e_{30} = \{b,c,e,d,a\},$$

$$e_{31} = \{b,d,a,c,e\}, e_{32} = \{b,d,a,e,c\}, e_{33} = \{b,d,c,a,e\}, e_{34} = \{b,d,c,e,a\}, e_{35} = \{b,d,e,a,c\},$$

$$e_{36} = \{b,d,e,c,a\}, e_{37} = \{b,e,a,c,d\}, e_{38} = \{b,e,a,d,c\}, e_{39} = \{b,e,c,a,d\}, e_{40} = \{b,e,c,d,a\},$$

$$e_{41} = \{b,e,d,a,c\}, e_{42} = \{b,e,d,c,a\}, e_{43} = \{c,d,a,b,e\}, e_{44} = \{c,d,a,e,b\}, e_{45} = \{c,d,b,a,e\},$$

$$e_{46} = \{c,d,b,e,a\}, e_{47} = \{c,d,e,a,b\}, e_{48} = \{c,d,e,b,a\}, e_{49} = \{c,e,a,b,d\}, e_{50} = \{c,e,a,d,b\},$$

$$e_{51} = \{c,e,b,a,d\}, e_{52} = \{c,e,b,d,a\}, e_{53} = \{c,e,d,a,b\}, e_{54} = \{c,e,d,b,a\}, e_{55} = \{d,e,a,b,c\},$$

$$e_{56} = \{d,e,a,c,b\}, e_{57} = \{d,e,b,a,c\}, e_{58} = \{d,e,b,c,a\}, e_{59} = \{d,e,c,a,b\}, e_{60} = \{d,e,c,b,a\}.$$

Elements of the form {a,b,{c,d,e}}:

$$f_1 = [a,b,c,d,e], f_2 = [a,b,c,e,d], f_3 = [a,b,d,e,c], f_4 = [a,c,b,d,e], f_5 = [a,c,b,e,d],$$

$$f_6 = [a,c,d,e,b], f_7 = [a,d,b,c,e], f_8 = [a,d,b,e,c], f_9 = [a,d,c,e,b], f_{10} = [a,e,b,c,d],$$

$$f_{11} = [a,e,b,d,c], f_{12} = [a,e,c,d,b], f_{13} = [b,c,a,d,e], f_{14} = [b,c,a,e,d], f_{15} = [b,c,d,e,a],$$

$$f_{16} = [b,d,a,c,e], f_{17} = [b,d,a,e,c], f_{18} = [b,d,c,e,a], f_{19} = [b,e,a,c,d], f_{20} = [b,e,a,d,c],$$

$$f_{21} = [b,e,c,d,a], f_{22} = [c,d,a,b,e], f_{23} = [c,d,a,e,b], f_{24} = [c,d,b,e,a], f_{25} = [c,e,a,b,d],$$

$$f_{26} = [c,e,a,d,b], f_{27} = [c,e,b,d,a], f_{28} = [d,e,a,b,c], f_{29} = [d,e,a,c,b], f_{30} = [d,e,b,c,a].$$

Every triple Tortken identity of degree 5 can be written as a linear combination of 60 and 30 multilinear monomials, that is:

$$\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_{60} e_{60} = 0$$

$$\mu_1 f_1 + \mu_2 f_2 + \cdots + \mu_{30} f_{30} = 0$$

We express every e_i and f_i in free Novikov algebra in terms of differential monomials and solve the system with respect to λ_i and μ_i. From solving these systems, we get an infinite set of solutions for λ and μ. We perform all our calculations using the software Wolfram Mathematica.

First we consider the first linear combination above. We will have 34 free unknowns. We fix a free unknown and substitute 1 for it and zero for the rest of the free unknowns. Then we calculate corresponding values of all other unknowns. That solution gives us a one polynomial triple identity in free Novikov algebra. We continue this procedure for all other free unknowns. We first exclude polynomial identities which are equivalent to others and have the following identities.

$\lambda_1 = 1$:

$$\{a,b,c,d,e\} - \{a,b,e,d,c\} - 2\{b,c,d,a,e\} - \{b,c,e,a,d\} + 3\{b,c,e,d,a\} - \{b,d,c,a,e\} + \{b,d,e,a,c\} + \{b,e,c,a,d\} + 2\{b,e,d,a,c\} - 3\{b,e,d,c,a\} - 2\{c,d,a,e,b\} + 2\{c,d,b,a,e\} + 2\{c,d,b,e,a\} + 4\{c,d,e,a,b\} - 6\{c,d,e,b,a\} - 3\{c,e,b,d,a\} + 3\{c,e,d,b,a\} + 2\{d,e,a,c,b\} - 2\{d,e,b,a,c\} + \{d,e,b,c,a\} - 4\{d,e,c,a,b\} + 3\{d,e,c,b,a\} = 0$$

$\lambda_2 = 1$:

$$\{a,b,c,e,d\} - \{a,b,e,d,c\} - \{b,c,d,a,e\} - 2\{b,c,e,a,d\} + 3\{b,c,e,d,a\} + \{b,d,e,a,c\} - \{b,d,e,c,a\} + 2\{b,e,d,a,c\} - 2\{b,e,d,c,a\} - \{c,d,a,e,b\} + \{c,d,b,a,e\} + 2\{c,d,e,a,b\} - 2\{c,d,e,b,a\} - \{c,e,a,d,b\} + \{c,e,b,a,d\} - \{c,e,b,d,a\} + 2\{c,e,d,a,b\} - \{c,e,d,b,a\} + 2\{d,e,a,c,b\} - 2\{d,e,b,a,c\} + \{d,e,b,c,a\} - 4\{d,e,c,a,b\} + 3\{d,e,c,b,a\} = 0$$

$\lambda_3 = 1$:

$$\{a,b,d,c,e\} - \{a,b,e,d,c\} - \{b,c,d,a,e\} + \{b,c,e,d,a\} - 2\{b,d,c,a,e\} + 2\{b,d,e,c,a\} + \{b,e,c,a,d\} + 2\{b,e,d,a,c\} - 3\{b,e,d,c,a\} - 2\{c,d,a,e,b\} + 2\{c,d,b,a,e\} + 2\{c,d,b,e,a\} + 4\{c,d,e,a,b\} - 6\{c,d,e,b,a\} + \{c,e,a,d,b\} - \{c,e,b,a,d\} - \{c,e,b,d,a\} - 2\{c,e,d,a,b\} + 3\{c,e,d,b,a\} + \{d,e,a,c,b\} - \{d,e,b,a,c\} - \{d,e,b,c,a\} - 2\{d,e,c,a,b\} + 3\{d,e,c,b,a\} = 0$$

$\lambda_4 = 1$:

$$\{a,b,d,e,c\} - \{a,b,e,d,c\} - \{b,d,c,a,e\} - \{b,d,e,a,c\} + 2\{b,d,e,c,a\} + \{b,e,c,a,d\} + \{b,e,d,a,c\} - 2\{b,e,d,c,a\} - \{c,d,a,e,b\} + \{c,d,b,a,e\} + 2\{c,d,e,a,b\} - 2\{c,d,e,b,a\} + \{c,e,a,d,b\} - \{c,e,b,a,d\} - 2\{c,e,d,a,b\} + 2\{c,e,d,b,a\} = 0$$

$\lambda_5 = 1$:

$$\{a,b,e,c,d\} - \{a,b,e,d,c\} - \{b,c,e,a,d\} + \{b,c,e,d,a\} + \{b,d,e,a,c\} - \{b,d,e,c,a\} - \{b,e,c,a,d\} + \{b,e,d,a,c\} - \{c,e,a,d,b\} + \{c,e,b,a,d\} + \{c,e,b,d,a\} + 2\{c,e,d,a,b\} - 3\{c,e,d,b,a\} + \{d,e,a,c,b\} - \{d,e,b,a,c\} - \{d,e,b,c,a\} - 2\{d,e,c,a,b\} + 3\{d,e,c,b,a\} = 0$$

$\lambda_7 = 1$:

$$\{a,c,b,d,e\} - \{a,c,e,d,b\} - 2\{b,c,d,a,e\} - \{b,c,e,a,d\} + 3\{b,c,e,d,a\} - 2\{b,e,a,d,c\} + 2\{b,e,c,a,d\} + 4\{b,e,d,a,c\} - 4\{b,e,d,c,a\} - 2\{c,d,a,e,b\} + \{c,d,b,a,e\} + 2\{c,d,b,e,a\} + 5\{c,d,e,a,b\} - 6\{c,d,e,b,a\} + 2\{c,e,a,d,b\} - \{c,e,b,a,d\} - 3\{c,e,b,d,a\} - 2\{c,e,d,a,b\} + 4\{c,e,d,b,a\} + 2\{d,e,a,c,b\} - 2\{d,e,b,a,c\} + 2\{d,e,b,c,a\} - 4\{d,e,c,a,b\} + 2\{d,e,c,b,a\} = 0$$

$\lambda_8 = 1$:

$$\{a,c,b,e,d\} - \{a,c,e,d,b\} - \{b,c,d,a,e\} - 2\{b,c,e,a,d\} + 3\{b,c,e,d,a\} - 2\{b,e,a,d,c\} + 2\{b,e,c,a,d\} + 4\{b,e,d,a,c\} - 4\{b,e,d,c,a\} - \{c,d,a,e,b\} + \{c,d,b,a,e\} + 3\{c,d,e,a,b\} - 3\{c,d,e,b,a\} + \{c,e,a,d,b\} - \{c,e,b,a,d\} - \{c,e,b,d,a\} + \{c,e,d,b,a\} + 2\{d,e,a,c,b\} - 2\{d,e,b,a,c\} + 2\{d,e,b,c,a\} - 4\{d,e,c,a,b\} + 2\{d,e,c,b,a\} = 0$$

$\lambda_9 = 1$:

$$\{a,c,d,b,e\} - \{a,c,e,d,b\} - \{b,c,d,a,e\} + \{b,c,e,d,a\} - \{b,e,a,d,c\} + \{b,e,c,a,d\} + 2\{b,e,d,a,c\} - 2\{b,e,d,c,a\} - 2\{c,d,a,e,b\} + 2\{c,d,b,e,a\} + 4\{c,d,e,a,b\} - 4\{c,d,e,b,a\} + 2\{c,e,a,d,b\} - \{c,e,b,a,d\} - \{c,e,b,d,a\} - 2\{c,e,d,a,b\} + 2\{c,e,d,b,a\} + \{d,e,a,c,b\} - \{d,e,b,a,c\} + \{d,e,b,c,a\} - 2\{d,e,c,a,b\} + \{d,e,c,b,a\} = 0$$

$\lambda_{10} = 1$:

$$\{a,c,d,e,b\} - \{a,c,e,d,b\} - \{c,d,a,e,b\} + \{c,d,e,a,b\} + \{c,e,a,d,b\} - \{c,e,d,a,b\} = 0$$

$\lambda_{11} = 1$:

$$\{a,c,e,b,d\} - \{a,c,e,d,b\} - \{b,c,e,a,d\} + \{b,c,e,d,a\} - \{b,e,a,d,c\} + \{b,e,c,a,d\} + 2\{b,e,d,a,c\} - 2\{b,e,d,c,a\} + \{c,d,e,a,b\} - \{c,d,e,b,a\} - \{c,e,b,a,d\} + \{c,e,b,d,a\} + \{c,e,d,a,b\} - \{c,e,d,b,a\} + \{d,e,a,c,b\} - \{d,e,b,a,c\} + \{d,e,b,c,a\} - 2\{d,e,c,a,b\} + \{d,e,c,b,a\} = 0$$

$\lambda_{13} = 1$:

$$\{a,d,b,c,e\} - \{a,d,e,c,b\} - 2\{b,d,c,a,e\} - \{b,d,e,a,c\} + 3\{b,d,e,c,a\} - 2\{b,e,a,d,c\} + 2\{b,e,c,a,d\} + 4\{b,e,d,a,c\} - 4\{b,e,d,c,a\} - 2\{c,d,a,e,b\} + \{c,d,b,a,e\} + 2\{c,d,b,e,a\} + 5\{c,d,e,a,b\} - 6\{c,d,e,b,a\} + 2\{c,e,a,d,b\} - 2\{c,e,b,a,d\} - 4\{c,e,d,a,b\} + 4\{c,e,d,b,a\} + 2\{d,e,a,c,b\} - \{d,e,b,a,c\} - \{d,e,b,c,a\} - 2\{d,e,c,a,b\} + 2\{d,e,c,b,a\} = 0$$

$\lambda_{14} = 1$:

$$\{a,d,b,e,c\} - \{a,d,e,c,b\} - \{b,d,c,a,e\} - 2\{b,d,e,a,c\} + 3\{b,d,e,c,a\} - 2\{b,e,a,d,c\} + 2\{b,e,c,a,d\} + 4\{b,e,d,a,c\} - 4\{b,e,d,c,a\} - \{c,d,a,e,b\} + \{c,d,b,a,e\} + 3\{c,d,e,a,b\} - 3\{c,d,e,b,a\} + 2\{c,e,a,d,b\} - 2\{c,e,b,a,d\} - 4\{c,e,d,a,b\} + 4\{c,e,d,b,a\} + \{d,e,a,c,b\} - \{d,e,b,a,c\} + \{d,e,b,c,a\} - \{d,e,c,b,a\} = 0$$

$\lambda_{15} = 1$:

$$\{a,d,c,b,e\} - \{a,d,e,c,b\} - \{b,d,c,a,e\} + \{b,d,e,c,a\} - \{b,e,a,d,c\} + \{b,e,c,a,d\} + 2\{b,e,d,a,c\} - 2\{b,e,d,c,a\} - 2\{c,d,a,e,b\} + 2\{c,d,b,e,a\} + 4\{c,d,e,a,b\} - 4\{c,d,e,b,a\} + \{c,e,a,d,b\} - \{c,e,b,a,d\} - 2\{c,e,d,a,b\} + 2\{c,e,d,b,a\} + 2\{d,e,a,c,b\} - \{d,e,b,a,c\} - 2\{d,e,c,a,b\} + \{d,e,c,b,a\} = 0$$

$\lambda_{16} = 1$:

$$\{a,d,c,e,b\} - \{a,d,e,c,b\} - \{c,d,a,e,b\} + \{c,d,e,a,b\}` + \{d,e,a,c,b\} - \{d,e,c,a,b\} = 0$$

$\lambda_{17} = 1$:

$$\{a,d,e,b,c\} - \{a,d,e,c,b\} - \{b,d,e,a,c\} + \{b,d,e,c,a\} - \{b,e,a,d,c\} + \{b,e,c,a,d\} + 2\{b,e,d,a,c\} - 2\{b,e,d,c,a\} + \{c,d,e,a,b\} - \{c,d,e,b,a\} + \{c,e,a,d,b\} - \{c,e,b,a,d\} - 2\{c,e,d,a,b\} + 2\{c,e,d,b,a\}$$

$$- \{d,e,b,a,c\} + 2\{d,e,b,c,a\} + \{d,e,c,a,b\} - 2\{d,e,c,b,a\} = 0$$

$\lambda_{19} = 1$:

$$\{a,e,b,c,d\} - \{a,e,d,c,b\} - 2\{b,e,a,d,c\} + 3\{b,e,d,a,c\} - \{b,e,d,c,a\} - \{c,e,b,a,d\} + 2\{c,e,b,d,a\} + \{c,e,d,a,b\} - 2\{c,e,d,b,a\} + 2\{d,e,a,c,b\} - \{d,e,b,a,c\} - \{d,e,b,c,a\} - 2\{d,e,c,a,b\} + 2\{d,e,c,b,a\} = 0$$

$\lambda_{20} = 1$:

$$\{a,e,b,d,c\} - \{a,e,d,c,b\} - 2\{b,e,a,d,c\} + \{b,e,c,a,d\} + 2\{b,e,d,a,c\} - \{b,e,d,c,a\} + \{c,e,a,d,b\} - \{c,e,b,a,d\} - \{c,e,d,a,b\} + \{c,e,d,b,a\} + \{d,e,a,c,b\} - \{d,e,b,a,c\} + \{d,e,b,c,a\} - \{d,e,c,b,a\} = 0$$

$\lambda_{21} = 1$:

$$\{a,e,c,b,d\} - \{a,e,d,c,b\} - \{b,e,a,d,c\} + 2\{b,e,d,a,c\} - \{b,e,d,c,a\} - \{c,e,a,d,b\} - \{c,e,b,a,d\} + 2\{c,e,b,d,a\} + 2\{c,e,d,a,b\} - 2\{c,e,d,b,a\} + 2\{d,e,a,c,b\} - \{d,e,b,a,c\} - 2\{d,e,c,a,b\} + \{d,e,c,b,a\} = 0$$

$\lambda_{22} = 1$:

$$\{a,e,c,d,b\} - \{a,e,d,c,b\} - \{c,e,a,d,b\} + \{c,e,d,a,b\} + \{d,e,a,c,b\} - \{d,e,c,a,b\} = 0$$

$\lambda_{23} = 1$:

$$\{a,e,d,b,c\} - \{a,e,d,c,b\} - \{b,e,a,d,c\} + \{b,e,c,a,d\} + \{b,e,d,a,c\} - \{b,e,d,c,a\} + \{c,e,a,d,b\} - \{c,e,b,a,d\} - \{c,e,d,a,b\} + \{c,e,d,b,a\} - \{d,e,b,a,c\} + 2\{d,e,b,c,a\} + \{d,e,c,a,b\} - 2\{d,e,c,b,a\} = 0$$

$\lambda_{25} = 1$:

$$\{b,c,a,d,e\} - 2\{b,c,d,a,e\} - \{b,c,e,a,d\} + 2\{b,c,e,d,a\} - \{b,e,a,d,c\} + \{b,e,c,a,d\} + 2\{b,e,d,a,c\} - 2\{b,e,d,c,a\} - \{c,d,a,e,b\} + \{c,d,b,a,e\} + \{c,d,b,e,a\} + 2\{c,d,e,a,b\} - 3\{c,d,e,b,a\} - 2\{c,e,b,d,a\} + 2\{c,e,d,b,a\} + \{d,e,a,c,b\} - \{d,e,b,a,c\} + \{d,e,b,c,a\} - 2\{d,e,c,a,b\} + \{d,e,c,b,a\} = 0$$

$\lambda_{26} = 1$:

$$\{b,c,a,e,d\} - \{b,c,d,a,e\} - 2\{b,c,e,a,d\} + 2\{b,c,e,d,a\} - \{b,e,a,d,c\} + \{b,e,c,a,d\} + 2\{b,e,d,a,c\} - 2\{b,e,d,c,a\} - \{c,d,a,e,b\} + \{c,d,b,a,e\} + 2\{c,d,e,a,b\} - 2\{c,d,e,b,a\} - \{c,e,b,d,a\} + \{c,e,d,b,a\} + \{d,e,a,c,b\} - \{d,e,b,a,c\} + \{d,e,b,c,a\} - 2\{d,e,c,a,b\} + \{d,e,c,b,a\} = 0$$

$\lambda_{28} = 1$:

$$\{b,c,d,e,a\} - \{b,c,e,d,a\} - \{c,d,b,e,a\} + \{c,d,e,b,a\} + \{c,e,b,d,a\} - \{c,e,d,b,a\} = 0$$

$\lambda_{31} = 1$:

$$\{b,d,a,c,e\} - 2\{b,d,c,a,e\} - \{b,d,e,a,c\} + 2\{b,d,e,c,a\} - \{b,e,a,d,c\} + \{b,e,c,a,d\} + 2\{b,e,d,a,c\} - 2\{b,e,d,c,a\} - \{c,d,a,e,b\} + \{c,d,b,a,e\} + \{c,d,b,e,a\} + 2\{c,d,e,a,b\} - 3\{c,d,e,b,a\} + \{c,e,a,d,b\} - \{c,e,b,a,d\} - 2\{c,e,d,a,b\} + 2\{c,e,d,b,a\} - \{d,e,b,c,a\} + \{d,e,c,b,a\} = 0$$

$\lambda_{32} = 1$:

$$\{b,d,a,e,c\} - \{b,d,c,a,e\} - 2\{b,d,e,a,c\} + 2\{b,d,e,c,a\} - \{b,e,a,d,c\} + \{b,e,c,a,d\} + 2\{b,e,d,a,c\} - 2\{b,e,d,c,a\} - \{c,d,a,e,b\} + \{c,d,b,a,e\} + 2\{c,d,e,a,b\} - 2\{c,d,e,b,a\} + \{c,e,a,d,b\} - \{c,e,b,a,d\} - 2\{c,e,d,a,b\} + 2\{c,e,d,b,a\} = 0$$

$\lambda_{37} = 1$:

$$\{b,e,a,c,d\} - \{b,e,a,d,c\} - \{b,e,c,a,d\} + \{b,e,d,a,c\} + \{c,e,b,d,a\} - \{c,e,d,b,a\} - \{d,e,b,c,a\} + \{d,e,c,b,a\} = 0$$

$\lambda_{43} = 1$:

$$\{c,e,a,b,d\} - \{c,e,a,d,b\} - \{c,e,b,a,d\} + \{c,e,b,d,a\} + \{c,e,d,a,b\} - \{c,e,d,b,a\} = 0$$

Here we have all the identities coming from the second linear combination involving the second type of bracketing:

$\mu_1 = 1$:

$$[a,c,b,d,e] - [a,c,d,e,b] - [a,d,b,c,e] + [a,d,c,e,b] - 1/2\ [a,e,b,c,d] + 1/2\ [a,e,b,d,c] - [b,c,a,d,e] + [b,c,d,e,a] + [b,d,a,c,e] - [b,d,c,e,a] + 1/2\ [b,e,a,c,d] - 1/2\ [b,e,a,d,c] + 1/2\ [c,e,a,d,b] - 1/2\ [c,e,b,d,a] - 1/2\ [d,e,a,c,b] + 1/2\ [d,e,b,c,a] = 0$$

$\mu_2 = 1$:

$$[a,c,b,e,d] - [a,c,d,e,b] - 3/2\ [a,e,b,c,d] + 1/2\ [a,e,b,d,c] + [a,e,c,d,b] - [b,c,a,e,d] + [b,c,d,e,a] + 3/2\ [b,e,a,c,d] - 1/2\ [b,e,a,d,c] - [b,e,c,d,a] + [c,d,a,e,b] - [c,d,b,e,a] - 1/2\ [c,e,a,d,b] + 1/2\ [c,e,b,d,a] - 1/2\ [d,e,a,c,b] + 1/2\ [d,e,b,c,a] = 0$$

$\mu_4 = 1$:

$$[a,d,b,e,c] - [a,d,c,e,b] - [a,e,b,d,c] + [a,e,c,d,b] - [b,d,a,e,c] + [b,d,c,e,a] + [b,e,a,d,c] - [b,e,c,d,a] + [c,d,a,e,b] - [c,d,b,e,a] - [c,e,a,d,b] + [c,e,b,d,a] = 0$$

$\mu_5 = 1$:

$$[a,b,c,d,e] - [a,b,d,e,c] - [a,d,b,c,e] + [a,d,c,e,b] - 1/2\ [a,e,b,c,d] + 3/2\ [a,e,b,d,c] - [a,e,c,d,b] - [b,c,a,d,e] + [b,c,d,e,a] + 1/3\ [b,d,a,c,e] + 4/3\ [b,d,a,e,c] - 5/3\ [b,d,c,e,a] + 1/6\ [b,e,a,c,d] - 5/6\ [b,e,a,d,c] + 2/3\ [b,e,c,d,a] + 2/3\ [c,d,a,b,e] - 4/3\ [c,d,a,e,b] + 2/3\ [c,d,b,e,a] + 1/3\ [c,e,a,b,d] + 5/6\ [c,e,a,d,b] - 7/6\ [c,e,b,d,a] - [d,e,a,b,c] + 1/2\ [d,e,a,c,b] + 1/2\ [d,e,b,c,a] = 0$$

$\mu_8 = 1$:

$$[a,b,c,e,d] - [a,b,d,e,c] - 3/2\ [a,e,b,c,d] + 3/2\ [a,e,b,d,c] - [b,c,a,e,d] + [b,c,d,e,a] + [b,d,a,e,c] - [b,d,c,e,a] + 1/2\ [b,e,a,c,d] - 1/2\ [b,e,a,d,c] + [c,e,a,b,d] - 1/2\ [c,e,a,d,b] - 1/2\ [c,e,b,d,a] - [d,e,a,b,c] + 1/2\ [d,e,a,c,b] + 1/2\ [d,e,b,c,a] = 0 .\ \blacksquare$$

Conclusion. This paper defines a triple Tortken product in Novikov algebras and gives a list of polynomial identities up to degree 5 satisfied by Tortken triple product in every Novikov algebra. We conclude that there is no identities of degree 3 and a list of identities of degree 5 given above satisfied by triple Tortken product in Novikov algebras over a field of characteristic zero.

Acknowledgments. The author gratefully acknowledges the many helpful suggestions of Professor N.A. Ismailov during the preparation of the paper.

References

1 Balinskii A.A., Novikov S.P. (1985) Poisson bracket of hamiltonian type, Frobenius algebras and Lie algebras, Dokladu AN SSSR, v. 283(5), pp.1036–1039.

2 Bremner M. (2018) On tortkara triple systems, Comm. Algebra, v. 46(6), pp. 2396–2404.

3 Dzhumadil'daev A.S. (2002) Novikov-Jordan algebras, Comm. Algebra,v. 30(11), pp. 5205–5240.

4 Dzhumadil'daev A.S. (2005) Special identity for Novikov-Jordan algebras Comm. Algebra, v. 33(5), pp. 1279–1287.

5 Dzhumadil'daev A.S. (2011) Codimension Growth and Non-Koszulity of Novikov Operad, Comm. Algebra, v. 39(8), pp. 2943–2952.

6 Dzhumadil'daev A.S., Löfwall C. (2002) Trees, free right-symmetric algebras, free Novikov algebras and identities, Homology, Homotopy and Appl.,4, no.2(1), pp.165–190.

7 Gelfand I.M., Dorfman I.Ya. (1979) Hamiltonian operators and related algebraic struc- tures, Func. Anal. Prilozhen, 13(4), pp. 13–30.

8 Jacobson N. (1949) Lie and Jordan triple systems, Amer. J. Math., 71, pp. 149–170.

9 Zhevlakov K.A., Slinko A.M., Shestakov I.P., Shirshov A.I. (1982) Rings That Are Nearly Associative, Academic Press, New York.

Information on the author

Mardanov Nurlybek Amangeldyuly
Master student, Kazakh-British Technical University, Tole bi str., 59, 050000, Almaty, Kazakhstan
ORCID ID: 0009-0002-5754-4212
E-mail: mardanov1602@gmail.com

Автор туралы мәлімет

Марданов Нұрлыбек Амангелдіұлы
Магистрант, Қазақстан-Британ техникалық университет, ул. Толе би, 59, 050000, г. Алматы, Казахстан
ORCID ID: 0009-0002-5754-4212
Email: mardanov1602@gmail.com

Информация об авторе

Марданов Нурлыбек Амангельдиевич
Магистрант, Казастанско-Британский технический университет, ул. Толе би, 59, 050000, г. Алматы, Казахстан
ORCID ID: 0009-0002-5754-4212
Email: mardanov1602@gmail.com