UDC 510.5
IRSTI 27.03.45
https://doi.org/10.55452/1998-6688-2023-20-2-36-42

Askarbekkyzy A.,[1*] Bazhenov N.A.[2]
[1]Al-Farabi Kazakh National University, Almaty, 050000, Kazakhstan
[2]Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia
[*]E-mail: ms.askarbekkyzy@gmail.com

INDEX SETS OF SELF-FULL LINEAR ORDERS ISOMORPHIC TO SOME STANDARD ORDERS

Abstract. The work of Bazhenov N.A., Zubkov M.V., Kalmurzayev B.S. started investigation of questions of the existence of joins and meets of positive linear preorders with respect to computable reducibility of binary relations. In the last section of this work, these questions were considered in the structure of computable linear orders isomorphic to the standard order of natural numbers. Then, the work of Askarbekkyzy A., Bazhenov N.A., Kalmurzayev B.S. continued investigation of this structure. In the last article, the notion of a self-full linear order played important role. A preorder R is called self-full, if for every computable function *g(x)*, which reduces *R* to *R*, the image of this function intersects all *supp(R)*-classes. In this article, we measure exact algorithmic complexities of index sets of all self-full recursive linear orders isomorphic to the standard order of natural numbers and to the standard order of integers. Researching the index sets allows us to measure exact algorithmic complexities of different notions in constructive structures, that we are investigating. We prove that the index set of all self-full computable linear orders isomorphic to the standard order of natural numbers is a Π^0_3-complete set. We also prove that the index set of all self-full computable linear orders isomorphic to the standard order of integers is Π^0_3-complete.

Key words: linear order, self-full order, index set, computable reducibility.

Асқарбекқызы А.,[1*] Баженов Н.А.[2]
[1]Әл-Фараби атындағы Қазақ Ұлттық Университеті, Алматы қ., 050000, Қазақстан
[2]С.Л. Соболев атындағы Математика Институты РҒА СБ, Новосибирск қ., 630090, Ресей
[*]E-mail: ms.askarbekkyzy@gmail.com

КЕЙБІР СТАНДАРТ РЕТТЕРГЕ ИЗОМОРФТЫ ӨЗІ ТОЛЫҚ СЫЗЫҚТЫ РЕТТЕРДІҢ ИНДЕКСТІ ЖИЫНДАРЫ

Аңдатпа. Баженов Н.А., Калмурзаев Б.С., Зубков М.В. жұмыстарында позитивті сызықтық реттердің бинарлы қатынастардың есептелімді көшірілуіне қатысты супремумы мен инфимумының бар болуы туралы сұрақтарын зерттеу басталған болатын. Жұмыстың соңғы тарауында бұл сұрақтар натурал сандардың стандартты ретіне изоморфты болатын есептелімді сызықты реттердің құрылымында қарастырылды. Одан кейін, Асқарбекқызы А., Баженов Н.А., Калмурзаев Б.С. жұмыстарында осы құрылымды зерттеу жалғасын тапты. Соңғы жұмыста өзітолық сызықты реттер ұғымы үлкен рөл ойнады. Егер *R* жарты ретін *R*-ге көшіретін кез келген есептелімді *g(x)* функциясының мәндер жиыны барлық *supp(R)*-класстарымен қиылысса, *R* жарты реті өзі толық рет деп аталады. Бұл мақалада натурал сандардың стандарт ретіне және бүтін сандардың стандарт ретіне изоморфты барлық сызықтық реттердің индексті жиындарының алгоритмдік күрделілігі нақты бағаланады. Индексті жиындарды зерттеу қарастырылып отырған конструктивті құрылымдардағы түрлі ұғымдардың нақты күрделілігін бағалауға мүмкіндік береді. Натурал сандардың стандартты ретіне изоморфты болатын өзітолық есептелімді сызықты реттердің индексті жиындары Π^0_3-толық жиын болатындығы дәлелденеді. Бүтін сандардың стандартты ретіне изоморфты болатын өзі толық есептелімді сызықты реттердің индексті жиындары Π^0_3-толық жиын болатындығы дәлелденеді.

Тірек сөздер: сызықты рет, өзітолық рет, индексті жиын, есептелімді көшіру.

Аскарбеккызы А.,[1*] Баженов Н.А.[2]
[1]Казахский Национальный университет им. аль-Фараби, г. Алматы, 050000, Казахстан
[2]Институт математики им. С.Л. Соболева СО РАН, г. Новосибирск, 630090, Россия
[*]E-mail: ms.askarbekkyzy@gmail.com

ИНДЕКСНЫЕ МНОЖЕСТВА САМОПОЛНЫХ ЛИНЕЙНЫХ ПОРЯДКОВ, ИЗОМОРФНЫХ НЕКОТОРЫМ СТАНДАРТНЫМ ПОРЯДКАМ

Аннотация. В работе Баженова Н.А., Зубкова М.В., Калмурзаева Б.С. было начато исследование вопросов существования супремумов и инфимумов позитивных линейных предпорядков относительно вычислимых сводимостей бинарных отношений, в последней главе эти вопросы были рассмотрены в структуре вычислимых линейных порядков, изоморфных стандартному порядку натуральных чисел. Далее, в работе Аскарбеккызы А., Баженова Н.А., Калмурзаева Б.С. было продолжено исследование этой структуры. В последней работе немаловажную роль сыграло понятие самополных линейных порядков. Предпорядок R называется самополным, если для любой вычислимой функции *g(x)*, осуществляющей сводимость R в R, ее область значений пересекает все supp(R)-классы. В данной статье оценивается точная алгоритмическая сложность индексных множеств всех самополных рекурсивных линейных порядков, изоморфных стандартному порядку натуральных чисел и целых чисел. Исследование индексных множеств позволяет оценить точную сложность различных понятий в исследуемых конструктивных структурах. Доказывается, что индексное множество самополных вычислимых линейных порядков, изоморфных стандартному порядку натуральных чисел, является Π^0_3-полным множеством. Доказывается, что индексное множество самополных вычислимых линейных порядков, изоморфных стандартному порядку целых чисел, является Π^0_3-полным множеством.

Ключевые слова: линейный порядок, самополный порядок, индексное множество, вычислимая сводимость

Introduction

In this paper, we investigate the algorithmic complexity of positive preorders on the set of natural numbers (we denote by ω) and isomorphic to the standard order of natural numbers (we denote this as ω_{st}), also to the standard order of integers (we denote this as ζ_{st}).

In this article, we keep the notations and terminology as in [1].

Let R, S be binary relations on ω. We say that R is *computably reducible* to S (denoted by $R \leq_c S$), if there is a computable function *f(x)* such that

$$x\,R\,y \leftrightarrow f(x)\,S\,f(y)$$

for all $x, y \in \omega$. We say that R and S are *computably equivalent* (denoted by $R \equiv_c S$), if $R \leq_c S$ and $S \leq_c R$ [].

There is a computable numbering α for the family of all positive preorders on ω. For simplicity, by P_i we will denote the positive preorder with index i (i.e., we will consider $\alpha(i) = P_i$). Given a class $K \subseteq \{P_x : x \in \omega\}$, we say that $I_K = \{x : P_x \in K\}$ is an *index set* of the class K. Studying index sets allows us to get an exact measure of the algorithmic complexity for different classes of positive preorders K. Article [3] was completely devoted to researching index sets for different classes of positive preorders.

The papers [4-9] measured index sets of classes of positive equivalence relations in universal numbering for the family of all positive equivalences.

Main provisions. Material and methods

Notation $\exists!\, x$ means that «there is a unique x», and $\exists^\infty x$ means that «there are infinitely many x». By *Id* we denote the identity equivalence relation on ω. By *card(A)* we denote the cardinality of a set A. Let R be a preorder and $j \in \omega$, then by $[j]_R$ we denote the set $\{x : (x, j) \in R \,\&(j, x) \in R\}$.

In proofs, we will use the following fact [10, Theorem 4.3.11]: $A \in \Sigma^0_3$ iff there is a computable ternary predicate R such that

- If $x \in A$, then there is a unique y, that $(\exists^\infty z)R(x,y,z)$;
- If $x \in A$, then for any $\neg(\exists^\infty z)\, R(x,y,z)$.

We will use the following notations:

Ceers $= (\{\deg_c(E)\colon E \text{ is a positive equivalence}\}; \leq_c)$,

Ceprs $= (\{\deg_c(P)\colon P \text{ is a positive preorder}\}; \leq_c)$,

Celps $= (\{\deg_c(L)\colon L \text{ is a positive linear preorder}\}; \leq_c)$.

Definition [7]. A preorder R is called *self-full* if for any reduction $g\colon R \leq_c R$ the following holds: for any number $j \in \omega$ there is $k \in \omega$, such that $g(k) \in [j]_R$.

The papers [11, 12] studied the following structure:

$\mathbf{\Omega} = (\{\deg_c(L)\colon L \text{ is a computable linear order isomorphic to } \omega_{st}\}; \leq_c)$

Moreover, in [11] it was proved that there exists an antichain of self-full degrees above any given degree $\boldsymbol{a} \in \mathbf{\Omega}$. This fact implies that the structure $\mathbf{\Omega}$ has continuum many automorphisms. Also, in [10] it was proved that there is no strong minimal cover for a non-self-full degree. This result implies that, in the structure $\mathbf{\Omega}$, the self-full degrees are precisely those elements that have a strong minimal cover. Note that a linear order $L \in \mathbf{\Omega}$ is self-full iff every computable function $f \neq id$ does not reduce L to L.

In this paper, we measure the index sets of the following classes:

- $I_{SF\Omega} = \{i\colon P_i \text{ is a self-full linear order isomorphic to } \omega_{st}\}$;
- $I_{SFZ} = \{i\colon P_i \text{ is a self-full linear order isomorphic to } \zeta_{st}\}$.

In particular, we prove that both sets are Π^0_3-complete sets.

Results and discussion

The index set $I_\Omega = \{i\colon P_i \in \Omega\}$ is Π^0_3-complete. The upper bound is implied by the following: $P_i \in \Omega$ is equivalent to this condition:

$$P_i \text{ is linear } \& \ P_i \text{ is antisymmetric } \& \ (\forall x)(\exists y)(\forall z > y)[x <_{P_i} z]. \quad (1)$$

The predicate $x <_{p_i} z$ is computable, if P_i is linear and antisymmetric. This means that the predicate (1) is equivalent to a Π^0_3-sentence.

The lower bound for I_Ω follows from Theorem 3(a) in [13], or from Example 2 in [14].

Theorem 1. The index set $I_{SF\Omega} = \{i\colon P_i \text{ is a self-full linear order isomorphic to } \omega_{st}\}$ is Π^0_3-complete.

Proof. Self-fulness of P_i is equivalent to the following:

$$(\forall e)\Big[\exists y(\varphi_e(y)\uparrow) \vee \Big[(\forall u,v)\big[u \leq_{P_i} v \leftrightarrow \varphi_e(u) \leq_{P_i} \varphi_e(v)\big] \rightarrow (\forall a \exists b)\big(a \sim_{P_i} \varphi_e(b)\big)\Big]\Big],$$

which is equivalent to a Π^0_3-condition. From the proof given above, $P_i \in \Omega$ is equivalent to a Π^0_3-condition. Then, the conjunction of these conditions is also equivalent to a Π^0_3-condition.

Now we show the completeness. Suppose that a set A belongs to the class Π^0_3. Then there is a computable relation $Q(x,y,z)$ such that

$$x \notin A \leftrightarrow (\exists! y)(\exists^\infty z)Q(x,y,z)$$

For every $e \in \omega$ we satisfy the following requirements for the constructed order $L = L_x$:

- SF_e: if $\varphi_e \neq id$, then φ_e does not reduce L to L.
- IS_e: if $(\exists^\infty z)Q(x,e,z)$, then $L \notin \Omega$.

In the construction of relation L there will be conflicts between strategies, which will be resolved by finite injury priority. Assume that the set of SF-strategies is linearly ordered of type ω: for instance, $SF_0 < SF_1 < \cdots$. A given strategy IS_e does not conflict with the other strategies, so IS_e essentially will work in the background mode.

In the strategy SF_e, we say that $2x_e$ is a fresh number if $2x_e$ is greater than $\varphi_{e'}(a)\downarrow$ for the higher priority strategies $SF_{e'}$, $e' < e$. In the construction, a notion "putting a fresh number after some number α in the order L_x at some stage s+1" means that we define $L_{x,s+1} = L_{x,s} \cup \{(z,y): a <_{L_{x,s}} y\} \cup \{(y,z): y \leq_{L_{x,s}} a\}$, where z is the least odd number not in $dom(L_{x,s})$.

STRATEGY for SF_e:

1. Wait for a fresh number $2x_e$, such that $\varphi_e(2x_e)\downarrow>_L 2x_e$;
2. Let $a := \varphi_e(2x_e)$. Wait until $\varphi_e(a)$ is defined;
3. If $\varphi_e(a)\downarrow>_L a$, then put $k := card([a,\varphi_e(a)]_L)$ fresh numbers after $2x_e$. Every time when $card([a,\varphi_e(a)]_L)$ increases, we will put a fresh number after $2x_e$.

Strategy SF_e has two outcomes:

wait: Stuck at step 1 or step 2. Then one of the following is true:

(a) The function φ_e is not total.

(b) $\varphi_e(2x_e)\downarrow\leq_L 2x_e$, or $\varphi_e(a)\downarrow\leq_L a$. Then φ_e = id or φ_e does not reduce L to L.

In both cases the requirement SF_e is satisfied.

act: Reaching step 3. If $\varphi_e(a)\leq_L a$, then the requirement is satisfied. Otherwise, $card([2x_e;a]_L) > card([a;\varphi_e(\bar{a})]_L)$. Then φ_e cannot be a computable reduction from L to L, and in this case we initialize the lower priority strategies of type SF_j, which chose $2x_j$ less than $\varphi_e(\alpha)$, as its own number at step 1. "The strategy SF_e is initialized" means that this strategy starts again from its step 1.

STRATEGY for IS_e:

1. Choose the number $2e$;
2. Let $k := card(\{z: Q(x,e,z)\})$. Put k fresh odd numbers after $2e$ in L.

CONSTRUCTION.

At stage 0, we assume that $L_{x,0} = \{(2z,2y): z \leq y\}$.

At stage $s+1$ we visit all strategies SF_i and IS_i for $i\leq s$. In IS_i we define k as $card(\{z: Q(x,i,z)\ \&\ z \leq s\})$. And in IS_i all conditions will be considered at the stage s.

Define $L_x = \cup_{s\in\omega} L_{x,s}$.

Lemma 1.1. L_x is a linear order on ω.

Proof. It is known that the function $f(x) = x+2$ has infinitely many Gödel numbers (let $\varphi_{e_i} = f$). Then $\varphi_{e_i}(2x_{e_i})\downarrow$ and strictly greater than $2x_{e_i}$ for every e_i. Moreover, $\varphi_{e_i}(2x_{e_i}) = 2x_{e_i} + 2 = a$, $\varphi_{e_i}(a)\downarrow$ and strictly greater than α. According to the strategy SF_{e_i}, at least one odd number should be put after $2x_{e_i}$. Since there are infinitely many such e_i, each odd number will be put after some number. Hence, L_i is a linear order on ω.

Lemma 1.2. If $x \notin A$, then $L_x \ncong \omega$.

Proof. Let $x \notin A$, then $(\exists! y)(\exists^\infty z)Q(x,y,z)$. Inside the interval $[2y,2y+2]_{L_x}$, the construction builds ω^*. Hence, $L_x \ncong \omega$.

Lemma 1.3. If $x \notin A$, then $L_x \ncong \omega$.

Proof. Let z be a natural number. We show that only finitely many elements will be enumerated inside $[2z,2z+2]_{L_x}$. According to the strategy IS_e, by the construction precisely $k := card(\{z: Q(x,e,z)\})$ elements will be enumerated after the number $2z$ (i.e., finitely many elements).

This number $2z$ can be the fresh number $2x_e$ for some strategies SF_e. Let SF_{e_0} be the first strategy, which reached step 3. At the first stage s_0, when it happened, it is obvious that the interval $[a;\varphi_{e_0}(a)]_{L_{x,s_0}}$ is finite. Consequently, there are finitely many intervals $[2y,2y+2]_{L_x}$ in this interval. If new elements are enumerated inside the interval $[2y,2y+2]_{L_x}$ according to the strategy IS_y, then some elements after $2x_e$ could be enumerated.

In addition, another strategy with higher priority SF_e can pick $2z$ as its own number at step 1 and reach step 3. In this case, every action from above will be repeated. Since there are only finitely many strategies with higher priority, in the end only finitely many elements will be enumerated inside the interval $[2z, 2z+2]_{L_x}$.

Since z is an arbitrary number, the constructed order L_x will be isomorphic to ω.

The sequence $(L_x)_{x\in\omega}$ is a computable numbering of some subfamily of the family of all positive preorders. Consequently, there is a computable function f such that $L_x = P_{f(x)}$ for every $x \in \omega$.

$$x \in A \leftrightarrow f(x) \in I_{SF\Omega}.$$

Theorem 1 is proved.

For $i \in \omega$, the preorder P_i is a computable linear order isomorphic to ζ_{st} if and only if:

P_i is linear & P_i is antisymmetric &

$$\neg[\exists x\, \forall y\, (x \leq_{P_i} y)] \& \neg[\exists x\, \forall y\, (x \geq_{P_i} y)] \& (\forall x, y)(\exists z)$$
$$(\forall u)[x <_{P_i} u <_{P_i} y \rightarrow u < z] \tag{2}$$

The predicate $x <_{P_i} z$ is computable, if P_i is linear and antisymmetric. Hence, the predicate (2) is equivalent to a Π^0_3-sentence.

Theorem 2. The index set $I_{SFZ} = \{x: P_x \text{ is a self-full linear order isomorphic to } \zeta_{st}\}$ is Π^0_3-complete.

Proof. We take the order L_x constructed in Theorem 1, and we define the order Z_x as follows:

$$a \leq_{Z_x} b \leftrightarrow [a = 2k \,\&\, b = 2m \,\&\, k \leq_{L_x} m] \vee$$
$$[a = 2k+1 \,\&\, b = 2m+1 \,\&\, m \leq_{L_x} k] \vee$$
$$[a = 2k+1 \,\&\, b = 2m]$$

Suppose that L_x is a self-full order isomorphic to ω_{st}. We show self-fullness of Z_x by reduction to a contradiction. Assume that Z_x is a non-self-full order. Then there is a computable function $f(x) \neq id$ that reduces Z_x to Z_x. Let t be an element such that $f(t) \neq t$. We construct a function $g: \omega \rightarrow \omega$ as follows:

- if $f(t) <_{Z_x} t$, then

$$g(y) = \begin{cases} y, & 2y+1 >_{Z_x} t, \\ \dfrac{f(2y+1)-1}{2}, & 2y+1 \leq_{Z_x} t. \end{cases}$$

- if $t <_{Z_x} f(t)$, then

$$g(y) = \begin{cases} y, & 2y <_{Z_x} t, \\ \dfrac{f(2y)}{2}, & 2y \geq_{Z_x} t. \end{cases}$$

A simple analysis shows that the function $g(x) \neq id$ computably reduces L_x to L_x, which contradicts the self-fullness of L_x.

If $L_x \ncong \omega_{st}$, then it is obvious that $Z_x \ncong \zeta_{st}$. Thus, reduction of an arbitrary Π^0_3-set F to I_{SFZ} can be proved as in Theorem 1.

Theorem 2 is proved.

Conclusion

Upper bound of the index sets of these classes was found:

- $I_{SF\Omega} = \{i: P_i$ is a self-full linear order isomorphic to $\omega_{st}\}$;
- $I_{SFZ} = \{i: P_i$ is a self-full linear order isomorphic to $\zeta_{st}\}$.

It is proved that both sets are Π^0_3-complete sets.

Acknowledgments

The work was supported by the Ministry of Science and Higher Education of the Republic of Kazakhstan, grant AP08856493 "Positive graphs and computable reducibility on them as mathematical model of databases."

REFERENCES

1 Ershov Yu.L. (1977) Theory of numberings. Moscow: Nauka. (In russian).

2 Bernardi C. and Sorbi A. (1983) Classifying positive equivalence relations, The Journal of Symbolic Logic, vol. 48, no. 3, pp. 529–538.

3 Kalmurzayev B.S., Bazhenov N.A. and Torebekova M.A. (2022) Ob indeksnyh mnozhestvah dlya klassov pozitivnyh predporyadkov, Algabra i logika, vol. 61, no.1, pp. 42–76.4 (In russian).

4 Gao S. and Gerdes P. (2001) Computably enumerable equivalence relations, Studia Logica: An International Journal for Symbolic Logic, vol. 67, no. 1, pp. 27–59.

5 Andrews U., Lempp S., Miller J.S., Ng K.M., San Mauro L. and Sorbi A. (2014) Universal computably enumerable equivalence relations, The Journal of Symbolic Logic, vol. 79, no. 1, pp. 60–88.

6 Andrews U. and Sorbi A. (2016) The complexity of index sets of classes of computably enumerable equivalence relations, The Journal of Symbolic Logic, vol. 81, no. 4, pp. 1375–1395.

7 Andrews U. and Sorbi A. (2019) Joins and meets in the structure of ceers, Computability, vol. 8, no. 3–4, pp. 193-241.

8 Andrews U. and Badaev S.A. (2020) On isomorphism classes of computably enumerable equivalence relations, The Journal of Symbolic Logic, vol. 85, no. 1, pp. 61–86.

9 Badaev S.A. and Sorbi A. (2016) Weakly precomplete computably enumerable equivalence relations, Mathematical Logic Quarterly, vol. 62, issue 1–2, pp. 111–127.

10 Soare R.I. (2016) Turing Computability, Theory and applications. Berlin: Springer.

11 Askarbekkyzy A., Bazhenov N.A. and Kalmurzayev B.S. (2022) Computable Reducibility for Computable Linear Orders of Type *ω*, Journal of Mathematical Sciences, vol. 267, pp. 429–443.

12 Bazhenov N.A., Kalmurzayev B.S. and Zubkov M.V. (2023) A note on joins and meets for positive linear preorders, Siberian Electronic Mathematical Reports, vol. 20, no. 1, pp. 1–16.

13 Kreisel, G., Shoenfield, J. and Wang, H. (1960) Number theoretic concepts and recursive well-orderings, Arch math Logik, vol. 5, 42–64.

14 Ash C.J. and Knight J.F. (1990) Pairs of recursive structures, Annals of Pure and Applied Logic, vol. 46, issue 3, pp. 211–234.

Information on the authors

Askarbekkyzy Aknur (corresponding author)
Master Student, Al-Farabi Kazakh National University, Al-Farabi Street, 71, Almaty, 050000, Kazakhstan
ORCID ID: 0000-0003-0075-4438
E-mail: ms.askarbekkyzy@gmail.com

Bazhenov Nikolay Alekseevich
Candidate of Physical and Mathematical Sciences, Senior Researcher at Sobolev Institute of Mathematics, 4, academician Koptyug av., 630090, Novosibirsk, Russia
ORCID ID: 0000-0002-5834-2770
E-mail: bazhenov@math.nsc.ru

Авторлар туралы мәліметтер

Асқарбекқызы Ақнұр (корреспонденция авторы)
Магистрант, Әл-Фараби атындағы Қазақ Ұлттық Университеті, Әл-Фараби к-сі, 71, Алматы қ., 050000, Қазақстан
ORCID ID: 0000-0003-0075-4438
E-mail: ms.askarbekkyzy@gmail.com

Баженов Николай Алексеевич
Физика-математика ғылымдарының кандидаты, аға ғылыми қызметкер, С.Л. Соболев атындағы Математика Институты РҒА СБ, Новосибирск қ., 630090, Ресей
ORCID ID: 0000-0002-5834-2770
E-mail: bazhenov@math.nsc.ru

Информация об авторах

Аскарбеккызы Акнур (автор для корреспонденции)
Магистрант, Казахский Национальный университет им. аль-Фараби, ул. Аль-Фараби, 71, г. Алматы, 050000, Казахстан
ORCID ID: 0000-0003-0075-4438
E-mail: ms.askarbekkyzy@gmail.com

Баженов Николай Алексеевич
Кандидат физико-математических наук, старший научный сотрудник Института математики им. С.Л. Соболева СО РАН, г. Новосибирск, 630090, Россия
ORCID ID: 0000-0002-5834-2770
E-mail: bazhenov@math.nsc.ru