UDC 510.5
IRSTI 27.03.45
https://doi.org/10.55452/1998-6688-2023-20-1-14-20

Nurlanbek D.D.

Al-Farabi Kazakh National University, 050000, Almaty, Kazakhstan
E-mail: nurlanbek.dias21@gmail.com

ON THE EXISTENCE OF UNIVERSAL NUMBERINGS

Abstract. The paper is devoted to research existence property of universal numberings for different computable families. A numbering α is reducible to a numbering β if there is computable function *f* such that $\alpha = \beta \circ f$. A computable numbering α for some family *S* is universal if any computable numbering β for the family *S* is reducible to α. It is well known that the family of all computably enumerable (c.e.) sets has a computable universal numbering. In this paper, we study families of almost all c.e. sets, recursive sets, and almost all differences of c.e. sets, namely questions about the existence of universal numberings for given families. We proved that there is no universal numbering for the family of all recursive sets. For families of c.e. sets without an empty set or a finite number of finite sets, there still exists a universal numbering. However, for families of all c.e. sets without an infinite set, there is no universal numbering. Also, we proved that family $\Sigma_2^{-1} \setminus B$ and the family Σ_1^{-1} has no universal Σ_2^{-1}-computable numbering for any $B \in \Sigma_2^{-1}$.

Key words: Computable numbering, computably enumerable sets, Rogers semilattices, Ershov's hierarchy, universal numbering.

Нұрланбек Д.Д.

Әл-Фараби атындағы Қазақ Ұлттық Университеті, 050000, Алматы қ., Қазақстан
E-mail: nurlanbek.dias21@gmail.com

УНИВЕРСАЛ НӨМІРЛЕУЛЕР ТАБЫЛАТЫНДЫҒЫ ТУРАЛЫ

Аңдатпа. Бұл мақала әр түрлі есептелімді үйірлердің универсал нөмірлеулері табылу қасиеттерін зерттеуге бағытталған. Қандайда бір α нөмірлеуі басқа бір β нөмірлеуіне көшіріледі деп аталынады. Егер $\alpha = \beta \circ f$ теңдігін қанағаттандыратын *f* есептелімді функциясы табылатын болса, кез келген *S* үйірі үшін α нөмірлеуі универсал болып табылады. Егер *S* үйіріндегі кез келген β нөмірлеуі α нөмірлеуіне көшірілетін болса, барлық рекурсив саналымды (р. с.) жиындар үйірінің универсал нөмірлеуі табылатындығы белгілі. Біз барлық дерлік жиындардың үйірлерін, рекурсив жиындардың және барлық дерлік р. с. жиындардың айырымының үйірлері үшін универсал нөмірлеулері табылатын табылмайтын қасиеттерін зерттейміз. Біз барлық рекурсив жиындардың үйірінде универсал нөмірлеу жоқ екенін дәлелдедік. Сондай-ақ, бос элементі жоқ, ақырлы жиындардың ақырлы саны жоқ р. с. жиындардың үйірінде универсал нөмірлеу болатынын көрсеттік. Ал ақырсыз р. с. жиыны жоқ барлық р. с. жиындардың үйіріне келетін болсақ, бұл жағдайда универсал нөмірлеу болмайтынын дәлелдедік. Сондай-ақ, біз $\Sigma_2^{-1} \setminus B$ және Σ_1^{-1} үйірлерінде кез келген $B \in \Sigma_2^{-1}$ үшін Σ_2^{-1}-есептелімді нөмірлеу болмайтынын дәлелдедік.

Тірек сөздер: есептелімді нөмірлеулер, рекурсив саналымды жиындар, Роджерс жарты торлары, Ершов иерархиясы, универсал нөмірлеулер.

Нурланбек Д.Д.

Казахский Национальный университет имени аль-Фараби, 050000, г. Алматы, Казахстан
E-mail: nurlanbek.dias21@gmail.com

О СУЩЕСТВОВАНИИ УНИВЕРСАЛЬНЫХ НУМЕРАЦИЙ

Аннотация. Данная статья посвящена исследованию свойства существования универсальных нумераций для различных семейств. Говорят, что нумерация α сводится к нумерации β, если существует вычислимая функция

f такая, что $\alpha = \beta \circ f$. Вычислимая нумерация α для некоторого семейства S универсальна, если любая вычислимая нумерация β для семейства S сводится к α. Хорошо известно, что семейство всех вычислимо перечислимых (в.п.) множеств имеет вычислимую универсальную нумерацию. В данной работе мы изучаем семейства почти всех в.п. множеств, рекурсивные множества и почти все разности в.п. множеств, а именно вопросы о существовании универсальных нумераций для данных семейств. Мы доказали, что для семейства всех рекурсивных множеств не существует универсальной нумерации. Также для семейств в.п. множества без пустого элемента, без конечного числа конечных множеств, все еще есть универсальная нумерация. Что касается семейств всех в.п. множества без бесконечного множества, то в этом случае универсальной нумерации не будет. Также мы доказываем, что семейство $\Sigma_2^{-1} \setminus B$ и семейство Σ_1^{-1} не имеют универсальной Σ_2^{-1}-вычислимой нумерации для любой $B \in \Sigma_2^{-1}$

Ключевые слова: вычислимые нумерации, вычислимо перечислимые множества, полурешетки Роджерса, иерархия Ершова, универсальная нумерация.

1. **Introduction**

The paper studies computable numberings for different families. In particular, we will investigate families for the existence of computable numberings and universal numberings.

The standard numberings of the family of all c.e. sets and of the family of all unary partial computable functions are denoted by $\{W_x\}_{x\in\omega}$ and $\{\varphi_x\}_{x\in\omega}$, respectively. A binary function $\langle x, y\rangle$ given by the rule

$$\langle x, y\rangle = \frac{(x+y)^2 + 3x + y}{2}$$

is a bijection of ω^2 onto ω, which is called *Cantor's pairing function*. By l and r we denote the uniquely defined functions such that $\langle l(x), r(x)\rangle = x$, $l(\langle x, y\rangle) = x$, and $r(\langle x, y\rangle) = y$ for all $x, y \in \omega$. For a finite set X we denote its cardinality by $card(X)$.

Let S be any countable set. By [1] any surjective mapping of the set of all natural numbers ω onto S is called *numbering* for the family S.

We say that numbering α is *computable* if the set $\{(x, n): x \in \alpha(n)\}$ is computably enumerable, and by $Com(S)$ we denote the set of all computable numberings for the family S. A family S is called computable if $Com(S)$ is non-empty.

A numbering α is *reducible to* β, if there exists a computable function f such that $\alpha(x) = \beta(f(x))$ for all $x \in \omega$, and we denote it as $\alpha \leq \beta$.

We say that numberings α, β are equivalent if $\alpha \leq \beta$ and $\beta \leq \alpha$, and denote it as $\alpha \equiv \beta$. By $deg(\alpha)$ we denote the set of all numberings which equivalent to α, i.e. $deg(\alpha) = \{\beta: \beta \equiv \alpha\}$. For family S the degree structure $(\{deg(\alpha): \alpha \in Com(S)\}, \leq)$ is called *Rogers semilattice* of S. It is important to note that if the Rogers semilattice has the largest element, then the numberings to this degree are called universal. That is, a computable numbering α of a family S is called *universal* if $\beta \leq \alpha$ for all computable numberings $\beta \in Com(S)$. More details about the properties of the classical Rogers semilattice can be found, for example, in [1-13].

Later, in [14], it was proposed to generalize the concepts of the Rogers semilattice for various computational classes. In this paper, we will be interested in generalizations for the Ershov hierarchy. Recall here that a set $A \subseteq \omega$ is in Ershov's hierarchy class Σ_n^{-1} if A is *n-computably enumerable* (*n*-c.e.), i.e., if $A = lim_s A_s$ for a uniformly computable sequence of functions A_s such that $A_0 = \emptyset$ and for each x, there are at most n many s such that $A_s(x) \neq A_{s+1}(x)$. Here, by $A_s(x)$ we denote the characteristic function for the set A_s. Furthermore, a set A is a *difference of computably enumerable sets* (*d*-c.e.) if A is 2-c.e., i.e., if A is of the form $A_0 \setminus A_1$ for computably enumerable sets A_0 and A_1.

2. **Main provisions. Material and methods.**

We call a numbering α for a family S is Σ_n^{-1}*-computable* (or equivalently, *n-computable*), if the relation $\{(x, n): x \in \alpha(n)\}$ is in Σ_n^{-1}. Note that if a family of sets S has a Σ_n^{-1}-computable numbering, then every set in S is in Σ_n^{-1}. For a family S by $Com_n^{-1}(S)$ we will denote the set of all Σ_n^{-1}-computable numberings for the family S. The quotient structure of $Com_n^{-1}(S)$ modulo equivalence of the numberings ordered by the relation reducibility of numberings is also called Rogers semilattice and denote it as $\mathcal{R}_n^{-1}(S)$. Similarly, a numbering $\alpha \in Com_n^{-1}(S)$ is called universal in $Com_n^{-1}(S)$ if $\beta \leq \alpha$ for all $\beta \in Com_n^{-1}(S)$.

The study of Rogers semilattices in the Ershov hierarchy is interesting because in it a number of unexpected results have been obtained. For example, it was shown in [15] that there is a family S consisting of just two d-c.e. sets such that $Com_2^{-1}(S)$ has no universal numbering, despite the fact that in classical numbering theory every finite family has a universal numbering. Khutoretskii's theorem states that the Rogers semilattice of any family of c.e. sets have either at most one or infinitely many elements [9]. Furthermore, a lemma used in the inductive step of the proof of this theorem demonstrates that, no Rogers semilattice can be partitioned into a principal ideal and a principal filter. But, Badaev and Lempp in [16] show that such decomposition is possible for some family of d-c.e. sets. The question of whether the full statement of Khutoretskii's Theorem fails for families of d-c.e. sets remains open. In view of the properties of the F. Stephan operator [17], it suffices to research Rogers semilattices for families of sets at two lower levels in the Ershov hierarchy. Other results on Rogers semilattices in Ershov hierarchy can be found, for example, in [18-26].

In section 2.1, we prove that the following families have no universal computable numberings: the family of all computable sets (theorem 1); for any infinite c.e. set A the family of all c.e. sets without set A (theorem 3). In case when A is a finite set, then the family of all c.e. sets without set A has a universal computable numbering (theorem 2). From this result in the set of all c.e. sets we can characterize finite sets in terms of Rogers semilattices. Namely, a c.e. set A is finite iff Rogers semilattice $\mathcal{R}_1^{-1}(\Sigma_1^{-1} \setminus A)$ has the greatest element (corollary 2). In section 2.2 we focused on Σ_2^{-1}-computable numberings. In particular, we prove that the following families have no universal Σ_2^{-1}-computable numberings: the family of all c.e. sets; for any d-c.e. set A the family $\Sigma_2^{-1} \setminus A$.

3. Results and Discussion

3.1. Computable numberings

This section provides the proofs of existence (or not) of the computable numberings and universal computable numberings for some families of c.e. sets. Let's denote by *Rec* the family of all computable sets.

Theorem 1. The family *Rec* is computable and *Rec* has no universal computable numbering.

Proof. First of all, we will prove that the family *Rec* has a computable numbering (see also [27]). We will construct, step by step, an approximation for numbering ν as follow: we present the construction for a fixed number e and additional, we will construct a computable function r.

Step 0. Assume that $\nu_0(e) = \emptyset$ and $r(0) = 0$.

Step s+1. If $\forall\ y \leq r(s)\ [\varphi_{e,s}(y) \downarrow\in \{0,1\}]$, then assume that

$$\nu_{s+1}(e) = \{x: x \leq r(s)\ \&\ \varphi_{e,s}(x) = 1\}\ and\ r(s+1) = r(s) + 1.$$

Otherwise, assume that $\nu_{s+1}(e) = \nu_s(e)$ and $r(s+1) = r(s)$.

The construction's description is complete.

Assume $\nu(x) = \cup_s \nu_s(x)$ for all $x \in \omega$.

Now we show that ν is a computable numbering for the family *Rec*. At first, from construction it is not hard to see that ν is computable numbering for some family, i.e. $\{(x,e): x \in \nu(e)\}$ is c.e.

Let's show that image of ν is *Rec*. If φ_e is characteristic function, i.e. $\{0,1\}$-valued total function, then obvious that $r(s)$ increases to infinite and $\nu(e)$ to be a set $\{x: \varphi_e(x) = 1\}$. That is $\nu(e)$ is computable set whose characteristic function is φ_e. If φ_e is not characteristic function, i.e. either $\{0,1\}$-valued nor total function, then there is minimal n, such that $\varphi_{e,s}(n) \notin \{0,1\}$ or $\varphi_e(n)$ is undefined. By construction function $r(s)$ do not increase more than n, so $\nu(e)$ remains finite.

For any recursive set A there is e such that $\nu(e) = A$. So, for the set A there is its characteristic recursive function φ_e. Since φ_e is $\{0,1\}$-valued total function, as discussed above $\nu(e)$ is a set which characteristic function is φ_e.

Now we will prove that for the family *Rec* there is no universal computable numbering.

Assume α be any computable numbering for the family *Rec*. Let $\{K_s\}_{s\in\omega}$ be a computable approximation for the halting problem. We will define numbering β as follow: for any $x \in \omega$

$$\beta(2x) = \alpha(x)$$

and define $\beta(2x+1)$ with the following construction

Step 0. Assume $\beta_0(2x+1)=\emptyset$ and $r(0)=0$.

Step s+1. If $\varphi_{x,s}(2x+1)\downarrow=y$ and $\beta_s(2x+1)\cap[0,r(s)]=\alpha_s(y)\cap[0,r(s)]$, then assume $\beta_{s+1}(2x+1)=K_s\cap[0,r(s)]$ and $r(s+1)=r(s)+1$.

Otherwise, $\beta_{s+1}(2x+1)=\beta_s(2x+1)$ and $r(s+1)=r(s)$.

The construction's description is complete. Assume $\beta(x)=\cup_s \beta_s(x)$.

Let's show that β is computable numbering for the family *Rec*. Since α is computable numbering and by construction clear that β is computable numbering for some family. It is clear that $\beta(2x)\in Rec$. Now, let *x* be any number. If $\varphi_x(2x+1)$ undefined, then by construction $\beta_s(2x+1)=\beta_0(2x+1)=\emptyset$ for all $s\in\omega$. So $\beta(2x+1)=\emptyset$ which belongs to *Rec*. If $\varphi_x(2x+1)$ is defined and equal to some *y*, then $\beta(2x+1)$ is finite set, because $\alpha(y)\neq K$. Really, let *m* be the least number such that *α(y)* and *K* different from each other. Then $\beta_s(2x+1)\cap[0,r(s)]\neq\alpha_s(y)\cap[0,r(s)]$ when $r(s)=m$. Function *r(s)* does not increase more than *m*, so $\beta(2x+1)$ remains finite.

Now, suppose that $\beta\leq\alpha$. Moreover, suppose φ_e is a recursive function which reduces β to α. It is clear that $\varphi_e(2e+1)\downarrow=y$ for some *y*. As discussed above there is m such that $\beta(2e+1)\cap[0,m]\neq\alpha(y)\cap[0,m]$. So $\beta(2e+1)\neq\alpha(\varphi_e(2e+1))$ which contradicts that φ_e is reduces β to α. Hence α can not be universal computable numbering. Theorem 1 is proved.

Now, we will show that there still be universal computable numbering for a family of all c.e. sets even if we remove any finite set.

Theorem 2. Let *F* be any finite set and *S* be a family of all c.e. sets without *F*. Then *S* has universal numbering.

Proof. By Friedberg's work [3], we know that there is Friedberg numbering for the family of c.e. sets. For instance, if we remove one element from the family, we still can have Friedberg numbering for a given family, we just enumerate them in other way.

The construction of computable universal numbering for the family *S* is split into two parts.

Case I. Assume that $F=\emptyset$. We construct infinitely many $\alpha_i(x)$ numberings that enumerates W_x with the $\{i\}$, i.e. $\alpha_n(x)=W_x\cup\{n\}$. Lets define the numbering β as follow:

$$\beta(\langle n,x\rangle)=\alpha_n(x).$$

It is not hard to see that β is computable numbering for the family *S*.

Now, let *ν* be a computable numbering for the family *S*. Since $S\subseteq\{W_i: i\in\omega\}$ there is computable function *f* such that $\nu(x)=W_{f(x)}$. Let ν_s be a computable approximation for the numbering *ν*. Lets define a function *h(x)* as follow:

$$h(x)=l(\mu_s[(l(s)\in\nu_s(x)])$$

Since $\nu(x)\neq\emptyset$ for any *x* the function *h* is total computable. So numbering *ν* is reducible to β via computable function $g(x)=\langle h(x),f(x)\rangle$. Indeed, since $h(x)\in\nu(x)=W_{f(x)}$ we know that $\alpha_{h(x)}(f(x))=W_{f(x)}\cup\{h(x)\}=W_{f(x)}$. Consequently, $\nu(x)=W_{f(x)}=\alpha_{h(x)}(f(x))=\beta(\langle h(x), f(x)\rangle)$. Which means that β is universal numbering for the family *S*.

Case II. Assume that $F\neq\emptyset$. For this case universal numbering β we can define as follows: for any *x* let $\beta(x)=\cup_s\beta_{s(x)}$, where $\beta_0(x)=\emptyset$ and

$$\beta_{s+1}(x)=\begin{cases}W_{x,s+1}, & if\ W_{x,s+1}\neq F;\\ \beta_s(x), & if\ W_{x,s+1}=F.\end{cases}$$

Again, let *ν* be a computable numbering for the family *S*. As in the previous case there is computable function *f* with $\nu(x)=W_{f(x)}$ for any *x*. Since $\nu(x)\neq F$ there is infinitely many *s* such that $W_{f(x),s}\neq F$. Then $\beta(f(x))=\nu(x)$. Theorem 2 is proved.

Corollary 1. If *S* is a family of all c.e. sets without finitely many finite sets, then *S* has universal numbering.

We can see the change if we remove some infinite set from the family *S* .

Theorem 3. Let F be any infinite c.e. set and S be a family of all c.e. sets without F. Then S has no universal numbering.

Proof. Proof of this theorem looks like proof of theorem 1. So as not to be repeated, we will only give the construction. Let F_s be a computable approximation for c.e. set F and α be a computable numbering for the family S. We will define numbering β as follow: for any $x \in \omega$

$$\beta(2x) = \alpha(x)$$

and define $\beta(2x+1)$ with the following construction

Step 0. Assume $\beta_0(2x+1) = \emptyset$ and $r(0) = 0$.

Step s+1. If $\varphi_{x,s}(2x+1) \downarrow = y$ and $\alpha_s(y) = \beta_s(2x+1)$, then assume $\beta_{s+1}(2x+1) = F_s \cap [0; r(s)]$ and $r(s+1) = r(s)+1$.

Otherwise, assume $\beta_{s+1}(2x+1) = \beta_s(2x+1)$ and $r(s+1) = r(s)$.

The construction description is complete. Let $\beta(x) = \cup_s \beta_s(x)$.

According to previous theorems, in the set of all c.e. sets we can define the notion of "finite sets" in terms of Rogers semilattices.

Corollary 2. W_e is finite set iff there is universal computable numbering for the family $\{W_i : i \in \omega\} \setminus \setminus \{W_e\}$.

3.2. $\mathbf{\Sigma_2^{-1}}$**-computable numberings**

In this section, we will use the following approximation for $\mathbf{\Sigma_2^{-1}}$-set.

Lemma 1. A set B is $\mathbf{\Sigma_2^{-1}}$ iff there is $\{0,1\}$-valued computable function $f(x,s)$ such that for all x, the following conditions is hold:

1. $B(x) = \lim_s f(x,s)$, with $f(x,0) = 0$;
2. $(\{s : f(x,s+1) \neq f(x,s)\}) \leq 2$

here, $B(x)$ is characteristic function for B. The function f is called $\mathbf{\Sigma_2^{-1}}$-approximation for the set B.

Theorem 4. Let B be a $\mathbf{\Sigma_2^{-1}}$-set. Then the family $S = \Sigma_2^{-1} \setminus \{B\}$ has no universal numbering in $Com_2^{-1}(S)$.

Proof. Let $\nu \in Com_2^{-1}(S)$ be any numbering. We will construct a numbering $\beta \in Com_2^{-1}(S)$ such that $\beta \not\leq \nu$. Let $f_B(x,s)$ be a $\mathbf{\Sigma_2^{-1}}$-approximation for B. Let $f_\nu(x,y,s)$ be a $\mathbf{\Sigma_2^{-1}}$-approximation for numbering ν. We define a $\mathbf{\Sigma_2^{-1}}$-approximation $f_\beta(x,y,s)$ for numbering β as follow: for all x,y,s assume

$$f_\beta(2x,y,s) = f_\nu(x,y,s),$$

and define $f_\beta(2x+1,y,s)$ with the following construction.

Step 0. Assume $f_\beta(2x+1,z,0) = 0$ for all x,z and $r(x,0) = 0$.

Step s+1. Let $x = l(s)$. If $\varphi_{x,s}(2x+1) \downarrow = y$ and $f_\beta(2x+1,z,s) = f_\nu(y,z,s)$ for all $z \leq r(x,s)$, then for all $z \leq r(x,s)$ set $f_\beta(2x+1,z,s+1) = f_B(z,s+1)$ and $r(x,s+1) = r(x,s)+1$.

Otherwise, assume $f_\beta(2x+1,z,s+1) = f_\beta(2x+1,z,s)$ and $r(x,s+1) = r(x,s)$.

Construction description is complete. Assume that, $\beta(x)(y) = \lim_s f_\beta(x,y,s)$.

It is not hard to see that $f_\beta(x,z,0) = 0$ for all $x,z,$ and $card(\{s : f_\beta(x,z,s+1) \neq f_\beta(x,z,s)\}) \leq 2$ because the function $\lambda s.f_\beta(2x+1,z,s)$ can change its value just because changes value $\lambda s.f_B(z,s)$.

In the case when B is finite the construction undergoes a few changes: assume that $f_B(x,s) = \chi_B(x)$ for all x,s. In this case at step 0 we start from the set ω instead $\emptyset$ (this means we assume $f_\beta(2x+1,z,0) = 1$ for all x,z.).

Now, suppose that $\mu \leq \nu$, then there is total computable function φ_e such that $\mu(x) = \nu(\varphi_e(x))$ for all x. On strength of $\mu \leq \nu$ via φ_e follows $\mu(2e+1) = \nu(y)$ for $y = \varphi_e(2e+1)$. Since for any z there is infinitely many s such that $f_\beta(2e+1,z,s) = f_\nu(y,z,s)$ the function $\lambda s.r(e,s)$ is increase to infinite, so

$$lim_s f_\nu(y,z,s) = lim_s f_\beta(2e+1,z,s) = lim_s f_B(z,s)$$

for all z. Hence $\nu(y) = B$. This is impossible, because v is numbering for family which does not contains the set B. Theorem 4 is proved.

Corollary 3. The family Σ_1^{-1} has no universal numbering in $Com_2^{-1}(\Sigma_1^{-1})$.

For proof the corollary 3 enough to take any proper B-c.e. set instead of B in the construction of the theorem 4.

3. **Conclusion**

In conclusion, we proved that there is no universal numbering for the family of all recusive sets. Also, for families of c.e. sets without an empty element, without a finite number of finite sets, there is still a universal numbering. As for the families of all c.e. sets without an infinite set, then in this case there will be no universal numbering. We proved that family $\Sigma_2^{-1} \setminus \{B\}$ and the family Σ_1^{-1} has no universal Σ_2^{-1}-computable numbering for any $B \in \Sigma_2^{-1}$.

4. **Acknowledgments**

The work of the authors is supported by the Ministry of Education and Science of the Republic of Kazakhstan, grant AP08856834 “Problems on Rogers semilattices of families of sets in the first and second levels of the Ershov hierarchy”.

REFERENCES

1 Yu. L. Ershov. (1977) Theory of Numerations, Nauka, Moscow. (In Russian)

2 Badaev S.A., Goncharov S.S. Theory of numberings: open problems. // Computability Theory and Its Applications, Amer. Math. Soc., Providences, 2000, pp. 23–38.

3 Friedberg, R.M. Three theorems on recursive enumeration. I. Decomposition. II. Maximal set. III. Enumeration without duplication. J. Symb. Log. 23, 309–316 (1958)/

4 Badaev S.A. On minimal enumerations, Siberian Adv. Math., 1992, v. 2, no. 1, pp. 1–30.

5 Badaev S.A. On cardinality of semilattices of numberings of non-discrete families, Sib. Math. J., 1993, v. 34, no. 5, pp. 795–800.

6 Badaev S.A. Minimal numberings of positively computable families/ Algebra and Logic, 1994, v. 33, no. 2, pp. 131–141.

7 Goncharov S.S., Badaev S.A. Families with one-element Rogers semi-lattice. Algebra and Logic, 1998, v. 37, no. 1, pp. 21–34.

8 Khutoretsky A.B. Two existence theorems for computable numerations. Algebra i Logika, 1969, v. 8, no. 4, pp. 484–492. (In Russian).

9 Khutoretsky A.B. On the cardinality of the upper semilattice of computable numberings, Algebra and Logic, 1971, v. 10, no. 5, pp. 348–352.

10 Rogers H. Godel numberings of partial computable functions. J. Symbolic Logic, 1958, v. 23, no. 3, pp. 49–57.

11 Selivanov V.L. Enumerations of families of general recursive functions. Algebra and Logic, 1976, v. 15, no. 2, pp. 128–141.

12 Selivanov V.L. Two theorems on computable enumerations. Algebra and Logic, 1976, v. 15, no. 4, pp. 297–306.

13 Badaev S.A., Goncharov S.S., Sorbi.A "Isomorphism types of Rogers semilattices for families from different levels of the arithmetical hierarchy. Algebra and Logic, 45:6 (2006), 361–370.

14 Goncharov S.S. and Sorbi A. Generalized computable numerations and nontrivial Rogers semilattices. Algebra and Logic, 36, no. 6, 359–369 (1997).

15 Abeshev K.Sh. On the existence of universal numberings for finite families of d.c.e. sets, Math. Log. Quart. 60, no. 3, 161–167 (2014).

16 Badaev S.A., Lempp S. A decomposition of the Rogers semilattice of a family of d.c.e. sets, The Journal of Symbolic Logic, v. 74, no 2, 2009.

17 I. Herbert, S. Jain, S. Lempp, M. Mustafa, and F. Stephan. Reductions between types of numberings, Ann. Pure Appl. Log., 170, no. 12 (2019), article 102716, pp. 1–25.

18 Abeshev K.Sh., Badaev S.A., Mustafa M. Families without minimal numberings, Algebra and Logic, v. 53, no 4, 2014.

19 Badaev S.A., Kalmurzayev B.S., Mukash N., Mustafa M. One-element Rogers semilattices in the Ershov hierarchy, Algebra and Logic, v. 60, no 4, 2021.

20 Badaev S.A., Mustafa M., Sorbi A. Friedberg numberings in the Ershov hierarchy, Arch. Math. Logic, v. 54, 2015.

21 Badaev S.A., Mustafa M., Sorbi A. Rogers semilattices of families of two embedded sets in the Ershov hierarchy, Mathematical Logic Quarterly, v. 58, no 4–5, 2012.

22 Badaev S.A. and Talasbaeva Zh.T. Computable numberings in the hierarchy of Ershov, Mathematical Logic in Asia, S. S. Goncharov (Ed.), World Scientific, NJ, 17–30 (2006).

23 Kalmurzayev B.S. Embeddability of the semilattice L_m^0 in Rogers semilattices, Algebra and Logic, v. 55, no 3, 2016.

24 Mustafa M., Sorbi A. Positive undecidable numberings in the Ershov hierarchy, Algebra and Logic, v. 50, no 6, 2012.

25 Ospichev S.S. Properties of numberings in various levels of the Ershov hierarchy, Journal of Mathematical Sciences, v. 188, no 4, 2013.

26 Ospichev S.S. Friedberg numberings in the Ershov hierarchy, Algebra and Logic, v. 54, no 4, 2015.

27 Odifreddi P. Classical Recursion Theory, Elsevier, Amsterdam, 1989.

Information about author

Nurlanbek Dias Daurenuly
Master student, Kazakh National University named after al-Farabi, Almaty, Kazakhstan.
ORCID ID: 0000-0002-1275-1413
E-mail: nurlanbek.dias21@gmail.com

Автор туралы мәліметтер

Нұрланбек Диас Дәуренұлы
Магистрант, әл-Фараби атындағы Қазақ Ұлттық Университеті, Алматы, Қазақстан.
ORCID ID: 0000-0002-1275-1413
E-mail: nurlanbek.dias21@gmail.com

Информация об авторе

Нурланбек Диас Дауренулы
Магистрант, Казахский Национальный университет имени аль-Фараби, Алматы, Казахстан.
ORCID ID: 0000-0002-1275-1413
E-mail: nurlanbek.dias21@gmail.com