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RANKS FOR FAMILIES OF REGULAR GRAPH THEORIES

Abstract. This article deals with families of regular graph theories. Using invariants of regular graph theory, a criterion for e-minimality, *a*-minimality, and α-minimality of subfamilies of the family of all regular graph theories is obtained. These ranks and degrees play a similar role for families of theories with hierarchies for definable theories, such as Morley's Hierarchies for a fixed theory, although they have their own peculiarities. The rank of families of theories can be thought of as a measure of the complexity or richness of these families. Thus, by increasing rank by expanding families, we produce richer families and get families with infinite rank, which can be considered "rich enough". The ranks for families of the theory of regular graphs with finite and infinite diagonals are described. The family of all regular graph theories has infinite rank. This follows from the fact that if a language consists of *m*-ary symbols, $m \geq 2$, then the family of all theories of the given language has an infinite rank. This also means that the family of all regular graph theories is not e-totally transcendental. The results obtained can be considered as a partial answer to the question posed in [5].

Key words: regular graph, rank, degree, family of theories, e-minimal family, *a*-minimal family, α-minimal family

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РАНГИ ДЛЯ СЕМЕЙСТВ ТЕОРИЙ РЕГУЛЯРНЫХ ГРАФОВ

Аннотация. В настоящей статье исследуются семейства теорий регулярных графов. Используя инварианты для теории регулярного графа, получены критерии e-минимальности, a-минимальности и α-минимальности подсемейств семейства всех теорий регулярных графов. Эти ранги и степени играют аналогичную роль для семейств теорий с иерархиями для определимых наборов теорий, таких как иерархии Морли для фиксированной теории, хотя они имеют собственные характеристики. Ранг семейств теорий, аналогичный рангу Морли, можно рассматривать как меру сложности или богатства этих семейств. Таким образом, повышая ранг за счет расширения семейств, мы производим более богатые семейства, получая семейства с бесконечным рангом, которые можно считать «достаточно богатыми». Описаны ранги для семейств теории регулярных графов с конечными и бесконечными диагоналями. Семейство всех теорий регулярных графов имеет бесконечный ранг. Это следует из факта, что если язык состоит из *m*-арных символов, $m \geq 2$, то семейство всех теорий данного языка имеет бесконечный ранг. Отсюда также следует, что семейство всех теорий регулярных графов не является e-тотально трансцендентным. Полученные результаты можно рассматривать как частичный ответ на вопрос, поставленный в работе [5].

Ключевые слова: регулярный граф, ранг, степень, семейство теорий, e-минимальное семейство, a-минимальное семейство, α-минимальное семейство

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ТҰРАҚТЫ ГРАФТАРДЫҢ ҮЙІРЛЕРІ ҮШІН РАНГІЛЕР ТЕОРИЯСЫ

Андатпа. Бұл мақалада тұрақты графтар теорияларының үйірлері зерттеледі. Тұрақты граф теориясы үшін инварианттарды пайдалана отырып, тұрақты графтардың барлық теориялары үйірлерінің е-минималдылығы, *a*-минималдылығы және α-минималдылығының критерийі алынады. Бұл рангтер мен дәрежелер теориялардың анықталатын жиындары үшін иерархиялары бар теориялар үйірлері үшін ұқсас рөл атқарады, мысалы, арнайы теория үшін Морли иерархиялары, бірақ олардың өзіндік сипаттамалары бар. Морли рангісіне ұқсас теориялар үйірлерінің рангісін осы үйірлердің күрделілігі немесе байлығының өлшемі ретінде қарастыруға болады. Осылайша, үйірлерді кеңейту және рангіні арттыру арқылы біз «жеткілікті бай» деп санауға болатын шексіз рангілі үйірлерді құрастыра аламыз. Ақырлы және ақырсыз диагональді тұрақты графтар теориялар үйірлері үшін рангілер сипатталған. Барлық тұрақты граф теорияларының үйірлері шексіз дәрежеге ие. Бұл келесі фактінің салдары: егер тіл m-ші ретті таңбаларынан тұрса, *m≥2*, онда берілген тілдегі барлық теориялар үйірлері шексіз дәрежеге ие болады. Бұл сонымен қатар барлық тұрақты граф теорияларының үйірлері тоталды трансценденттік емес екенін білдіреді. Алынған нәтижелерді [5] жұмыста қойылған сұраққа ішінара жауап ретінде қабылдауға болады .

Түйінді сөздер: тұрақты граф, ранг, дәреже, теориялар үйірі, е-минималды үйір, *a*-минималды үйір, α-минималды үйір

Introduction

Below we consider families $\mathcal{T}$ of consistent first-order theories in languages $\Sigma \subseteq \Sigma(\mathcal{T})$, where $\Sigma(\mathcal{T})$ is the set of language symbols that are represented in some sentences of some theories in $\mathcal{T}$. By $F(\Sigma)$ we denote the set of all formulas in the language Σ. If Σ is relational, i.e., it does not contain functional symbols, then we denote by $\mathcal{T}_\Sigma$ the family of all theories of the language Σ. For a sentence φ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} | \varphi \in T\}$. Any set $\mathcal{T}_\varphi$ is called the φ-*neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$

Main provisions

Definition [1]. A *regular graph* is a graph where all vertices have the same number of neighbors. A regular graph with vertices of degree *k* is called a *k-regular graph* or *regular graph of degree k*.

In [5], the ranks of complete theory families have been defined inductively as follows.

(1) For an empty family $\mathcal{T}$ we set the rank $RS(\mathcal{T}) = -1$ and for non-empty finite families $\mathcal{T}$ we set the rank $RS(\mathcal{T}) = 0$.

(2) For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we set $RS(\mathcal{T}) \geq \alpha$ if there are pairwise $\mathcal{T}$-inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that each φ_n-neighborhood has rank $RS(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.

(3) If α is a limit ordinal then $RS(\mathcal{T}) \geq \alpha$ if $RS(T) \geq \beta$ for any $\beta < \alpha$.

(4) We set $RS(\mathcal{T}) \geq \alpha$ if $RS(\mathcal{T}) \geq \alpha$ and $RS(\mathcal{T}) \geq \alpha + 1$.

(5) If $RS(\mathcal{T}) \geq \alpha$ for any α, we set $RS(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e-totally transcendental* if $RS(\mathcal{T})$ is an ordinal. If $\mathcal{T}$ is e-totally transcendental, we define the *degree* $ds(\mathcal{T})$ for the family $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $RS(\mathcal{T}_{\varphi_i}) = \alpha$.

Definition [4]. An infinite family $\mathcal{T}$ of complete theories is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, either $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.

Definition [5] A family $\mathcal{T}$, with infinitely many accumulation points, is called *a-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, either $\mathcal{T}_\varphi$ or $\mathcal{T}_{\neg\varphi}$ has finitely many accumulation points.

Let α be an ordinal. A family $\mathcal{T}$ of rank α is called α-*minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$ either $RS(\mathcal{T}_\varphi) < \alpha$ or $RS(\mathcal{T}_{\neg\varphi}) < \alpha$.

Materials and Methods

Denote by $\mathcal{T}_{reg}$ the family of all regular graph theories. For any theory $T \in \mathcal{T}_{reg}$, consider the pair (k, γ_k), where $k \in \omega, \gamma_k \in \omega \cup \{\infty\}$ is the number of k-regular connected components.

Let us first consider families with bounded γ_k.

In the case when for each theory T from the subfamily $\mathcal{T} \subset \mathcal{T}_{reg}$ it is true that the diameter d(T) is finite, as well as, the subfamily $\mathcal{T}$ is finite, therefore RS($\mathcal{T}$)=0, and the degree ds($\mathcal{T}$) is equal to the number of invariants. Let's illustrate how the degrees of families vary.

Let us deal with the finite family $\mathcal{T} \subset \mathcal{T}_{reg}$ consisting of theories $T_1, \dots, T_n$. If each theory has the same number of k_i-regular graphs, that is, $\gamma_{k_i} = s$ with $\gamma_{k_i} \neq \gamma_{k_j}$, $i \neq j$, then RS($\mathcal{T}$)=0, ds($\mathcal{T}$)=n, since $\mathcal{T}$ is represented as a disjoint union of finite subfamilies $\mathcal{T}_{\varphi_i} = \{T_i \in \mathcal{T} | \varphi_i \in T_i$ is a sentence describing k_i-regular graphs$\}$.

In the example above, one can notice that the degree of the family depends on the number of invariants. If for the theories considered in above we complicate the conditions that each theory has the same number of invariants, let, for example, r, then the degree of the family $\mathcal{T}$ is $ds(\mathcal{T}) = n \cdot r$. And if for different $r_1, \dots, r_n$, in each theory T_i there are r_i invariants, then the family $\mathcal{T}$ has degree $ds(\mathcal{T}) = \sum_{i=1}^{n} r_i$.

Results and Discussion

Proposition 2.1 (1) *A family $\mathcal{T}$ is e-minimal if and only if $\mathcal{T}$ is a set of regular graph theories with one arbitrary value $\gamma_k, \gamma_m = 0$ for $m \neq k$. (2) A family $\mathcal{T}$ is α-minimal if and only if $\mathcal{T}$ is a set of regular graph theories with two arbitrary values $\gamma_{k_1}, \gamma_{k_2}, \gamma_{k_0} = 0$ for $k_0 \neq k_1 \neq k_2$.*

(3) For countable ordinal α a family $\mathcal{T}$ is α-minimal if and only if $\mathcal{T}$ is a set of regular graph theories with arbitrary values $\gamma_{k_1}, \gamma_{k_2}, \dots, \gamma_{k_n}$, $n \in \omega$, $\gamma_{k_0} = 0$ for $k_0 \neq k_i \neq k_j$.

Proof. (1) It is known that if the family is finite, then the rank is 0, and the degree is equal to the number of invariants. If the family is infinite, then the number of accumulation points is considered. By the definition of e-minimality, either $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite. Moreover, an e-minimal family $\mathcal{T}$ has rank 1 and degree 1. This means that the family T has a single accumulation point

$$\bar{T} = \{\varphi \in \Sigma(\bar{\mathcal{T}}) | \mathcal{T}_\varphi \ is \ infinite\}.$$

In turn, the family consists of a finite number of theories with one arbitrary value $\gamma_m, \gamma_k = 0$, $m \neq k$, and an infinite number of theories with $\gamma_k, \gamma_m = 0$ for $m \neq k$.

The family $\mathcal{T}$ which is a set of regular graph theories with one arbitrary value $\gamma_m, \gamma_k = 0$, for $m \neq k$, consists of theories T_m with $m \in \omega \cup \{\infty\}$ k-regular graphs and the theory T_∞ with an infinite number of k-regular graphs. The theory T_∞ is the unique accumulation point for $\mathcal{T}$. Thus, RS($\mathcal{T}$) = 1, ds($\mathcal{T}$) = 1 and therefore the family $\mathcal{T}$ is e-minimal.

(2) Let the family $\mathcal{T}$ be not *a*-minimal. Then by [5, Proposition 2.13] RS($\mathcal{T}$) $\neq$ 2 and ds($\mathcal{T}$) $\neq$ 1. From the previous case, a family with rank 1 and degree 1 has a single accumulation point and only one invariant v_k. If RS($\mathcal{T}$) > 2 and ds($\mathcal{T}$) >1, then the family can be divided by the sentence φ into subfamilies $\mathcal{T}_\varphi$ and $\mathcal{T}_{\neg\varphi}$ with an infinite number of accumulation points and the family will have at least three invariants.

We are dealing with k_1-regular and k_2-regular graphs. Then we get a countable number of variants $(\gamma_{k_1}, \gamma_{k_2})$. Thus, there are countably many k_1-regular and k_2-regular graphs, forming the family $\mathcal{T}$. Here, each subfamily with one infinite γ_{k_1} or γ_{k_2} has RS = 1, and the only accumulation point with

$\gamma_{k_2} = \gamma_{k_2} = \infty$, has infinitely many k_l-regular, infinitely many k_2-regular graphs and RS= 2. Thus, for a given family $\mathcal{T}$, we obtain RS($\mathcal{T}$) = 2 and ds($\mathcal{T}$) = 1. Therefore, the family is α-minimal.

(3) It is proved as a right direction proof of the previous case, assuming that the family is not α -minimal. The family will either have a finite rank or an infinite (uncountable) rank. Hence the number of invariants is finite or uncountable.

If there is a countable number of k_i-regular graphs, $i \in \omega$, having a countable number of variants $(\gamma_{k_1}, \gamma_{k_2}, \ldots, \gamma_{k_i})$ one can construct an α-minimal family $\mathcal{T}$ consisting of a countable number of e-minimal subfamilies $\mathcal{T}_i$, $i \in \omega$. By the definition of α-minimality, the family $\mathcal{T}$ has RS($\mathcal{T}$) = α, ds($\mathcal{T}$) = 1 and is represented as $\mathcal{T}_{\wedge_{i\in\omega}\varphi_i}$.

The Proposition 2.1 has been proven.

Realizations of e-minimal, α-minimal, α-minimal subfamilies of the family $\mathcal{T}_{reg}$ of all regular graph theories show that it is possible to construct a subfamily $\mathcal{T}$ having a given countable rank and degree. Subfamily $\mathcal{T}$ is represented as a disjoint union of subfamilies $\mathcal{T}_{\varphi_1}, \ldots, \mathcal{T}_{\varphi_n}$, for some pairwise inconsistent sentences $\varphi_1, \ldots, \varphi_n$, such that each $\mathcal{T}_{\varphi_i}$ is α-minimal.

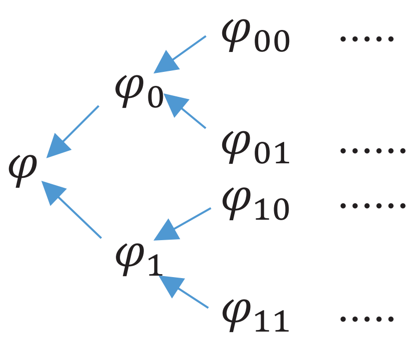
$RS(\mathcal{T}) = \alpha$		$\mathcal{T}_{\varphi_1} \sqcup \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \sqcup \mathcal{T}_{\varphi_n}$
	$\mathcal{T}_1$	$\mathcal{T}_{(1,1)} = \{T_{(1,1)} \ldots T_{(1,\infty)}\}, \ldots, \mathcal{T}_{(1,n)} = \{T_{(n,1)} \ldots T_{(n,\infty)}\}$
	. .	. . . .
	$\mathcal{T}_k$	$\mathcal{T}_{(k,1)} = \{T_{(1,1)} \ldots T_{(1,\infty)}\}, \ldots, \mathcal{T}_{(k,n)} = \{T_{(n,1)} \ldots T_{(n,\infty)}\}$
	. .	. . . .
	$\mathcal{T}_\alpha$	$\mathcal{T}_{(\alpha,1)} = \{T_{(1,1)} \ldots T_{(1,\infty)}\}, \ldots, \mathcal{T}_{(\alpha,n)} = \{T_{(n,1)} \ldots T_{(n,\infty)}\}$

An illustration of a family with a given countable rank and natural degree.

We now consider families of theories of regular graphs with infinite diameter and an unbounded number of γ_k. The next result shows that the family $\mathcal{T}_{reg}$ of all regular graph theories is not e-totally transcendental.

Theorem 2.2. $RS(\mathcal{T}_{reg}) = \infty$.

Proof. Repeating the arguments of [4, Proposition 4.4] and [2, Proposition 2.5] we can construct a 2-tree of sentences $\varphi, \varphi_0, \varphi_1, \varphi_{01}, \ldots$ indicating an infinite rank.



As noted in [2], the theorem also holds for acyclic regular graphs.

Conclusion

This article examines families of regular graph theories. By using invariants for the theory of regular graphs, a criterion for the e-minimality, the *a*-minimality and the α-minimality of the subfamilies of the family of all the theories of regular graphs is obtained. These ranks and degrees play a similar role for families of theories with hierarchies for definable theories, such as Morley's Hierarchies [3] for a fixed theory, although they have their own characteristics. The rank of families of theories, similar to Morley's rank, can be considered as a measure of the complexity or the richness of these families. Thus, by increasing the rank by enlarging the families, we produce richer families and obtain families of infinite rank which can be considered "rich enough". The family of all regular graph theories has infinite rank. This follows from the fact that if a language consists of m-ary symbols, $m \geq 2$, the family of all theories in the given language has infinite rank. This also implies that the family of all regular graph theories is not e-totally transcendental.

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