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ON (p, q)-SPLITTING FORMULAS IN ALMOST OMEGA-CATEGORICAL WEAKLY O-MINIMAL THEORIES

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Abstract. *The present paper concerns the notion of weak o-minimality introduced by M. Dickmann and originally studied by D. Macpherson, D. Marker, and C. Steinhorn. Weak o-minimality is a generalization of the notion of o-minimality introduced by A. Pillay and C. Steinhorn in series of joint papers. As is known, the ordered field of real numbers is an example of an o-minimal structure. We continue studying properties of almost omega-categorical weakly o-minimal theories. Almost omega-categoricity is a notion generalizing the notion of omega-categoricity. Recently, a criterion for binarity of almost omega-categorical weakly o-minimal theories in terms of convexity rank has been obtained. Binary convexity rank is the convexity rank in which parametrically definable equivalence relations are replaced by ∅ - definable equivalence relations. (p, q)-splitting formulas express a connection between non-weakly orthogonal non-algebraic 1-types in weakly o-minimal theories. In many cases, the binary convexity ranks of non-weakly orthogonal non-algebraic 1-types are not equal. The main result of this paper is finding necessary and sufficient conditions for equality of the binary convexity ranks for non-weakly orthogonal non-algebraic 1-types in almost omega-categorical weakly o-minimal theories in terms of (p, q)-splitting formulas.*

Keywords: *weak o-minimality, almost omega-categoricity, (p, q)-splitting formula, convexity rank, weak orthogonality, equivalence relation.*

ДЕРЛІК ОМЕГА-КАТЕГОРИЯЛЫҚ ӘЛСІЗ О-МИНИМАЛДЫ ТЕОРИЯЛАРЫНДА (p, q)-СЕКАТОРЛАР ТУРАЛЫ

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Аңдатпа. *Бұл жұмыс М. Дикманн енгізген және бастапқыда Д. Макферсон, Д. Маркер және Ч. Стейнхорн зерттеген әлсіз о-минималдылық түсінігіне қатысты. Әлсіз о-минималдылық – бұл А. Пиллай мен Ч. Стейнхорнның бірлескен мақалалар сериясында енгізген о-минималдылығы ұғымының жалпылауы. Белгілі болғандай, нақты сандардың реттелген өрісі о-минималды құрылымның алгебралық мысалы болып табылады. Біз дерлік омега категориялық әлсіз о-минималды теориялардың қасиеттерін зерттеуді жалғастырамыз. Омега-категориялық дерлік – бұл омега категориялық түсінігін жалпылайтын ұғым. Жақында дөңестік рангісі бойынша дерлік омега категориялық әлсіз о-минималды* *теориялардың бинарлық критерийі алынды. Бинарлық дөңестік рангісі – параметрлік анықталатын эквиваленттік қатынастар бос жиынмен анықталатын эквиваленттік қатынастармен ауыстырылатын дөңестік рангісі. (p, q)-бөлу формулалары әлсіз o-минималды теориялардағы әлсіз ортогональды алгебралық емес 1-түрлер арасындағы байланысты білдіреді. Көптеген жағдайларда әлсіз емес ортогональды алгебралық емес 1-түрлердің бинарлық дөңестік рангілері тең емес. Бұл жұмыстың негізгі нәтижесі (p, q)-бөлу формулалары тұрғысынан дерлік омега категориялық әлсіз o-минималды теориялардағы әлсіз ортогональды алгебралық емес 1-түрлері үшін екілік дөңес рангтарының теңдігі үшін қажетті және жеткілікті шарттарды табу болып табылады.*

Түйінді сөздер: *әлсіз o-минималдылық, дерлік омега-категориялық, (p, q)-бөлу формуласы, дөңестік ранг, әлсіз ортогональдық, эквиваленттік қатынас.*

О (p, q)-СЕКАТОРАХ В ПОЧТИ ОМЕГА-КАТЕГОРИЧНЫХ СЛАБО О-МИНИМАЛЬНЫХ ТЕОРИЯХ

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Аннотация. *Настоящая статья касается понятия слабой o-минимальности, введенного М. Дикманном и первоначально исследованного Д. Макферсоном, Д. Маркером и Ч. Стейнхорном. Слабая o-минимальность является обобщением понятия o-минимальности, введенного А. Пиллэем и Ч. Стейнхорном в серии совместных статей. Как известно, упорядоченное поле вещественных чисел является примером o-минимальной структуры. Мы продолжаем изучение свойств почти омега-категоричных слабо o-минимальных теорий. Почти омега-категоричность – это понятие, обобщающее понятие омега-категоричности. Недавно был получен критерий бинарности почти омега-категоричных слабо o-минимальных теорий в терминах ранга выпуклости. Бинарный ранг выпуклости – это ранг выпуклости, в котором параметрически определимые отношения эквивалентности заменяются пусто-определимыми отношениями эквивалентности. (p, q)-секаторы выражают связь между не слабо ортогональными неалгебраическими 1-типами в слабо o-минимальных теориях. В большинстве случаев бинарные ранги выпуклости не слабо ортогональных неалгебраических 1-типов не совпадают. Основным результатом данной статьи является нахождение необходимых и достаточных условий равенства бинарных рангов выпуклости для не слабо ортогональных неалгебраических 1-типов в почти омега-категоричных слабо o-минимальных теориях в терминах (p, q)-секаторов.*

Ключевые слова: *слабая o-минимальность, почти омега-категоричность, (p, q)-секатор, ранг выпуклости, слабая ортогональность, отношение эквивалентности.*

Preliminaries

Let L be a countable first-order language. Throughout this paper we consider L-structures and suppose that L contains a binary relation symbol $<$ which is interpreted as a linear order in these structures. A subset A of a linearly ordered structure M is convex if for all $\alpha, b \in A$ and $c \in M$ whenever $< \alpha < b$ we have $c \in A$. This paper concerns the notion of *weak o-minimality* that was initially deeply studied by D. Macpherson,

D. Marker, and C. Steinhorn in [1]. A *weakly o-minimal structure* is a linearly ordered structure $M = \langle M, <, \dots \rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in M. We recall that such a structure M is said to be *o-minimal* if any definable (with parameters) subset of M is a union of finitely many intervals and points in M. Thus, weak o-minimality generalizes the notion of o-minimality. Real closed fields with a proper convex valuation ring provide an important example of weakly o-minimal (not o-minimal) structures [2, 3].

Let A and B be arbitrary subsets of a linearly ordered structure M. Then the expression $A < B$ means that $\alpha < b$ whenever $\alpha \in A$ and $b \in B$, and $A < b$ means that A $< \{b\}$. For a subset A of M we introduce the following notations: $A^{+} := \{b \in M \mid A < b\}$ and $A^{-} := \{b \in M \mid b < A\}$. For an arbitrary one-type p we denote by $p(M)$ the set of realizations of p in M. If $B \subseteq M$ and E is an equivalence relation on M then we denote by B/E the set of equivalence classes (E-classes) which have representatives in B. If f is a function on M, then we denote by (f) the domain of M. A theory T is said to be *binary* if every formula of f is equivalent in T to a Boolean combination of formulas with at most two free variables.

Further throughout the paper we consider an arbitrary complete theory T (if unless otherwise stated), where M is a sufficiently saturated model of T.

Definition 1.1 Let T be a weakly o-minimal theory, $M \vDash T, A \subseteq M, p, q \in S_1(A)$ be non-algebraic. We say that p is not weakly orthogonal to q (denoting this by $p \not\perp^{w} q$) if there exist an L_A-formula $H(x, y)$, $\alpha \in p(M)$ and $\beta_1, \beta_2 \in q(M)$ such that $\beta_1 \in H(M, \alpha)$ and $\beta_2 \notin H(M, \alpha)$.

In other words, p is weakly orthogonal to q (denoting this by $p \perp^{w} q$) if $p(x) \cup q(y)$ has a unique extension to a complete 2-type over A.

Lemma 1.2 [4] Let T be a weakly o-minimal theory, $M \vDash T$, $A \subseteq M$. Then the relation of non-weak orthogonality is an equivalence relation on $S_1(A)$.

The definition of convexity rank for a set was introduced in [5].

Definition 1.3 Let T be a weakly o-minimal theory, M be a sufficiently saturated model of the theory T, A ⊆ M. The convexity rank of the set A (RC(A)) is defined as follows:

1) RC(A) = –1 if A = ∅.

2) RC(A) = 0 if A is finite and non-empty.

3) RC(A) ≥ 1 if A is infinite.

4) RC(A) ≥ α + 1 if there exists a parametrically definable equivalence relation E(x, y) such that there are $b_i \in$ A, i ∈ ω, which satisfy the following:

• For any i, j ∈ ω, whenever i ≠ j we have M ⊨ ¬E(b_i, b_j);

• For every i ∈ ω RC(E(M, b_i)) ≥ α and E(M, b_i) is a convex subset of A.

5) RC(A) ≥ δ if RC(A) ≥ α for all α < δ (δ is limit).

If RC(A) = α for some α, we say that RC(A) is defined. Otherwise (i.e. if RC(A)) ≥ α for all α), we put RC(A) = ∞.

The rank of convexity of a formula $\varphi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\varphi(M, \bar{a})$. The rank of convexity of a 1-type p is defined as the rank of convexity of the set $p(M)$, i.e., $RC(p) := RC(p(M))$. In particular, a theory has convexity rank 1 if there are no definable (with parameters) equivalence relations with infinitely many infinite convex classes.

We say that the convexity rank of an arbitrary set A is binary and denote it by $RC_{bin}(A)$ if parametrically definable equivalence relations are replaced by ∅-definable (i.e., binary) equivalence relations.

Definition 1.4 [6, 7] Let T be a complete theory, and $p_1(x_1), \dots, p_n(x_n) \in S_1(\emptyset)$. A type $q(x_1, \dots, x_n) \in S_n(\emptyset)$ is said to be a $(p_1, \dots, p_n)$-type if

$$q(x_1, \dots, x_n) \supseteq p_1(x_1) \cup \dots \cup p_n(x_n).$$

The set of all $(p_1, \dots, p_n)$-types of the theory T is denoted by $S_{p_1,\dots,p_n}(T)$. A countable theory T is said to be almost ω-categorical if for any types $p_1(x_1), \dots, p_n(x_n) \in S_1(\emptyset)$ there are only finitely many types $q(x_1, \dots, x_n) \in S_{p_1,\dots,p_n}(T)$.

Almost omega-categoricity is closely connected with the notion of Ehrenfeuchtness of a theory. So, in [6] it was proved that if T

is an almost omega-categorical theory with $I(T,\omega)=3$ then a dense linear order is interpreted in T. Nonetheless there exists an example (constructed by M.G. Peretyat'kin in [8]) of a theory with the condition $I(T,\omega)=3$ that is not almost omega-categorical.

In [9] the authors established almost omega-categoricity of Ehrenfeucht quite o-minimal theories and that the Exchange Principle for algebraic closure holds in almost omega-categorical quite o-minimal theories. Recently in [10], orthogonality of any family of pairwise weakly orthogonal non-algebraic 1-types over $\emptyset$ for such theories and binarity of almost omega-categorical quite o-minimal theories were proved. Also, in [11], binarity of almost omega-categorical weakly o-minimal theories of convexity rank 1 was established. At last, in the work [12], a criterion for binarity of almost omega-categorical weakly o-minimal theories in terms of convexity rank was found.

Theorem 1.5 [10] Let T be an almost omega-categorical weakly o-minimal theory, $p\in S_1(\emptyset)$ be non-algebraic. Then $RC_{bin}(p)<\omega$.

Recall some notions originally introduced in [1]. Let $Y\subset M^{n+1}$ be an $\emptyset$-definable subset, let $\pi\colon M^{n+1}\to M^n$ be the projection which drops the last coordinate, and let $Z:=\pi(Y)$. For each $\bar a\in Z$ let $Y_{\bar a}:=\{y:(\bar a,y)\in Y\}$. Suppose that for every $\bar a\in Z$ the set $Y_{\bar a}$ is convex and bounded above but does not have a supremum in M. We let $\sim$ be the $\emptyset$-definable equivalence relation on M^n given by

$\bar a\sim\bar b$ for all $\bar a,\bar b\in M^n\setminus Z$, and
$\bar a\sim\bar b\Leftrightarrow\sup Y_{\bar a}=\sup Y_{\bar b}$ if $\bar a,\bar b\in Z$.

Let $\bar Z:=Z/{\sim}$, and for each tuple $\bar a\in Z$ we denote by $[\bar a]$ the $\sim$-class of $\bar a$. There is a natural $\emptyset$-definable total order on $M\cup\bar Z$, defined as follows. Let $\bar a\in Z$ and $c\in M$. Then $\bar a<c$ if and only if $<c$ for all $w\in Y_{\bar a}$. Also, we say $c<[\bar a]$ iff $\neg([\bar a]<c)$, i.e. there exists $w\in Y_{\bar a}$ such that $c\le w$. If $\bar a$ is not $\sim$-equivalent to $\bar b$ then there is some $x\in M$ such that $[\bar a]<x<[\bar b]$ or $[\bar b]<x<\bar a$, and so $<$ induces a total order on $M\cup\bar Z$. We call such a set $\bar Z$ a sort (in this case, $\emptyset$-definable sort) in $\bar M$, where $\bar M$ is the Dedekind completion of M, and view $\bar Z$ as naturally embedded in $\bar M$. Similarly, we can obtain a sort in $\bar M$ by considering infima instead of suprema.

Thus, we will consider definable functions from M to its Dedekind completion $\bar M$, more precisely in definable sorts of the structure $\bar M$, representing infima or suprema of definable sets.

Let $A,D\subseteq M$, D be infinite, $Z\subseteq\bar M$ be an A-definable sort and $f:D\to Z$ be an A-definable function. We say f is locally increasing (locally decreasing, locally constant}) on D if for any element $a\in D$ there is an infinite interval $J\subseteq D$ containing $\{a\}$ so that f is strictly increasing (strictly decreasing, constant) on J; we also say f is locally monotonic on D if it is locally increasing or locally decreasing on D.

Let f be an A-definable function on $D\subseteq M$, E be an A-definable equivalence relation on D. We say f is strictly increasing (decreasing) on D/Eif for any $a,b\in D$ with $a<b$ and $\neg E(a,b)$ we have $f(a)<f(b)$ ($f(a)>f(b)$).

Proposition 1.6 [13] Let M be a weakly o-minimal structure, $A\subseteq M, p\in S_1(A)$ be a non-algebraic type. Then any A-definable function of which the domain contains the set $p(M)$ is locally monotonic or locally constant on $p(M)$.

Definition 1.7 [14] Let T be a weakly o-minimal theory, $M\vDash T$, $A\subseteq M$, $p\in S_1(A)$ be non-algebraic.

(1) An L_A -formula $F(x,y)$ is said to be p -preserving (or p-stable) if there exist elements $\alpha,\gamma_1,\gamma_2\in p(M)$ such that

$$[F(M,\alpha)\setminus\{\alpha\}]\cap p(M)\neq\emptyset \text{ and}$$
$$\gamma_1<F(M,\alpha)\cap p(M)<\gamma_2.$$

(2) A p-preserving formula $F(x,y)$ is said to be convex-to-right (left) if there exists an element $\alpha\in p(M)$ such that $F(M,\alpha)\cap p(M)$ is convex, α is the left (right) endpoint of the set $p(M)$ and $\alpha\in F(M,\alpha)$.

Definition 1.8 [15] Let $F(x,y)$ be a p -preserving convex-to-right (left) formula. We say that $F(x,y)$ is said to be equivalence-generating if for any $\alpha,\beta\in p(M)$ such that $M\vDash F(\beta,\alpha)$ the following holds:

$$M\vDash\forall x\big[x\ge\beta\to[F(x,\alpha)\leftrightarrow F(x,\beta)]\big]$$
$$(resp.\, M\vDash\forall x\big[x\le\beta\to[F(x,\alpha)\leftrightarrow F(x,\beta)]\big].$$

Lemma 1.9 [15] Let T be a weakly o-minimal theory, $M \vDash T, A \subseteq M, p \in S_1(A)$ be non-algebraic. Suppose that $F(x,y)$ is a p-preserving convex-to-right (left) formula that is also equivalence-generating. Then

(1) $G(x,y) := F(y,x)$ is a p-preserving convex-to-left (right) formula which is also equivalence-generating.

(2) $E(x,y) := F(x,y) \vee F(y,x)$ is an equivalence relation on $p(M)$ partitioning it into infinitely many infinite convex classes.

Proposition 1.10 [10] Let T be an almost omega-categorical weakly o-minimal theory, $p \in S_1(\emptyset)$ be non-algebraic. Then any p-preserving convex-to-right (left) formula is equivalence-generating.

In this work we present a criterion for equality of the binary convexity ranks for non-weakly orthogonal non-algebraic 1-types in almost omega-categorical weakly o-minimal theories.

Results

Recall the notion of a (p,q)-splitting formula introduced in [16] for non-algebraic isolated 1-types. Let $A \subseteq M, p, q \in S_1(A)$ be non-algebraic, p is not weakly orthogonal to q. Extending the definition of (p,q)-splitting formula to non-isolated case, we say that an L_A -formula $\varphi(x,y)$ is a (p,q)-splitting formula, if there is $a \in p(M)$ such that

$$\varphi(a,M) \cap q(M) \neq \emptyset,\ \neg\varphi(a,M) \cap q(M) \neq \emptyset,$$
$$\varphi(a,M) \cap q(M) \text{ is convex, and}$$
$$[\varphi(a,M) \cap q(M)]^- = [q(M)]^-.$$

If $\varphi_1(x,y)$, $\varphi_2(x,y)$ are (p,q)-splitting formulas then we say that $\varphi_1(x,y)$ is not *less than* $\varphi_2(x,y)$ if there is $a \in p(M)$ such that $\varphi_1(a,M) \cap q(M) \subseteq \varphi_2(x,y) \cap q(M)$. We say that (p,q)-25-splitting formulas $\varphi_1(x,y)$ and $\varphi_2(x,y)$ are equivalent ($\varphi_1(x,y) \sim \varphi_2(x,y)$) if $\varphi_1(x,y) \cap\ q(M) = \varphi_2(x,y) \cap q(M)$f or some (any) $a \in p(M)$.

Obviously, if $p, q \in S_1(A)$ are non-algebraic and p is not weakly orthogonal to q, then there is at least one (p,q)-splitting formula, and the set of all (p,q)-splitting formulas is partitioned into a linearly ordered set of equivalence classes with respect to ~. For every (p,q)-splitting formula $\varphi(x,y)$ we will consider the function f^{φ}, where $f^{\varphi} := \sup \varphi(x,M)$. Also, obviously that for any (p,q)-splitting formula $\varphi(x,y)$ the function f^{φ} is not constant on $p(M)$.

We will also say [17] that for a (p,q)-splitting formula $\varphi(x,y)$ the set $Rangef^{\varphi}{}_{p(M)}$ is everywhere dense in $q(M^{eq})$ if for any $b_1, b_2 \in q(M)$ with $b_1 < b_2$ there exists $a \in p(M)$ such that $b_1 < f^{\varphi}(a) < b_2$.

Example 2.1. Let $M = \langle M; <, P_1^1, P_2^1, E^2,$ be a linearly ordered structure so that M is a disjoint union of interpretations of unary predicates P_1 and P_2 with $P_1(M) < P_2(M)$. We identify the interpretation of P_1 with $\mathbb{Q}$, ordered as usual, and the interpretation of P_2 with $\mathbb{Q} \times \mathbb{Q}$, ordered lexicographically. The relation E is an equivalence relation on $P_2(M)$:

$$E((a_1,a_2),(c_1,c_2)) \Leftrightarrow a_1 = c_1\ for\ any\ (a_1,a_2),(c_1,c_2) \in P_2(M)$$

The relation R is defined as follows:

$$R(a,(b_1,b_2)) \Leftrightarrow b_1 \leq a\ for\ any\ a \in P_1(M), (b_1,b_2) \in P_2(M).$$

It is not difficult to establish that M is a countably categorical weakly o-minimal structure. Let $p := \{P_1(x)\}, q := \{P_2(x)\}$. Obviously, p and q determine complete types over $\emptyset$, p is not weakly orthogonal to q, $RC(p) = 1$, $RC(q) = 2$and $R(x,y)$ is a (p,q) -splitting formula. The function f^R is strictly increasing on $p(M)$, and the set $Range\, f^R{}_{p(M)}$ is not everywhere dense in $q(M^{eq})$.

Theorem 2.2 Let T be an almost omega-categorical weakly o-minimal theory, M be a sufficiently saturated model of $T, p, q \in S_1(\emptyset)$ be non-algebraic, p is not weakly orthogonal to q. Suppose that there exists an $\emptyset$-definable equivalence relation $E(x,y)$ partitioning $p(M)$ into infinitely many infinite convex classes. Then the following conditions are equivalent:

(1) $RC_{bin}(p) = RC_{bin}(q) + RC_{bin}(E(a,M))$ for some (any) $a \in p(M)$;

(2) $RC_{bin}(p) > RC_{bin}(q)$;

(3) for any (p,q)-splitting formula $R(x,y)$ there exists an $\emptyset$-definable equivalence relation $E'(x,y)$ partitioning $p(M)$ into infinitely many infinite convex classes so that f^R is constant

on each E'-class and the set $Range\, f^R{}_{p(M)}$ is everywhere dense in $q(M^{eq})$;

(4) for any (p,q)-splitting formula $R(x,y)$ the function f^R is constant on each E-class, the set $Range\, f^R{}_{p(M)}$ is everywhere dense in $q(M^{eq})$, and $E'(x,y)$ is maximal with this property.

Proof of Theorem 2.2. Let for a definiteness $RC_{bin}(p) = n$. Then there exist $\emptyset$-definable equivalence relations $E_1(x,y), \ldots, E_{n-1}(x,y)$ partitioning $p(M)$ into infinitely many infinite convex classes so that $E_1(a,M) \subset \cdots \subset E_{n-1}(a,M)$ for some (any) $a \in p(M)$. Obviously, by the hypotheses of the theorem $n \geq 2$.

(1) ($\Rightarrow$) (2). Obviously, since each E-class is infinite, i.e. $RC_{bin}\big(E(a,M)\big) \geq 1$.

(2) ($\Rightarrow$) (3). Suppose that $RC_{bin}(p) >$ $> RC_{bin}(q)$. Assume the contrary: there exists a (p,q)-splitting formula $R(x,y)$ such that for any $\emptyset$-definable equivalence relation $E'(x,y)$ partitioning $p(M)$ into infinitely many infinite convex classes $f^R(x) \coloneqq \sup R(x,M)$ is not constant on each E'-class. Then f^R is not constant on each E_1-class. But then f^R must be strictly monotonic (strictly increasing or strictly decreasing) on each E_1-class. Indeed, f^R can not be locally monotonic (non-strictly monotonic) on each E_1-class, since otherwise an $\emptyset$-definable equivalence relation $E_0(x,y)$ partitioning $p(M)$ into infinitely many infinite convex classes is appeared, so that $E_0(a,M) \subset E_1(a,M)$ for some (any) $a \in p(M)$ which contradicts the hypothesis that $E_1(x,y)$ is minimal among $\emptyset$-definable non-trivial equivalence relations on $p(M)$. Thus, f^R is strictly monotonic on each E_1-class. If the set $Range\, f^R{}_{p(M)}$ is not everywhere dense in $q(M^{eq})$, then there exist $b_1, b_2 \in q(M)$ such that $b_1 < b_2$ and for any $a \in p(M)$ either $f^R(a) \leq b_1$ or $b_2 \leq f^R(a)$. If f^R is strictly increasing on each E_1-class, then consider the following formula:

$$S(x,b_1) := \; b_1 \leq x \land \exists u\big[f^R(u) \leq b_1 \land$$
$$\land\, \forall t\big(u < t \land E_1(u,t) \to x < f^R(t)\big)\big].$$

If f^R is strictly decreasing on each E_1-class, then consider the following formula:

$$S(x,b_1) := \; b_1 \leq x \land \exists u\big[f^R(u) \leq b_1 \land$$
$$\land\, \forall t\big(t < u \land E_1(u,t) \to x < f^R(t)\big)\big].$$

It not difficult to see that $S(x,y)$ is a q-preserving convex-to-right formula. Then by almost omega-categoricity of T it must be equivalence-generating, whence we also have a contradiction with the fact that $E_1(x,y)$ is minimal among $\emptyset$-definable non-trivial equivalence relations on $p(M)$.

Further we consider the behaviour of the function f^R on each $E_2(a,M)/E_1$, where $a \in p(M)$. It must be strictly monotonic on each $E_2(a,M)/E_1$ and $Range\, f^R{}_{p(M)}$ must be everywhere dense in $q(M^{eq})$, since otherwise an $\emptyset$-definable equivalence relation $E''(x,y)$ is appeared with the property $E_1(a,M) \subset E''(a,M) \subset E_2(a,M)$ which contradicts the fact that E_2 is an immediate successor of E_1 among all $\emptyset$-definable equivalence relations on $p(M)$. Similarly, it can be proved that f^R is strictly monotonic on each $E_{k+1}(a,M)/E_k$, where $1 \leq k \leq n-2$ and f^R is strictly monotonic on $p(M)/E_{n-1}$.

Consider the following formulas:

$$E'_1(x,y) \coloneqq \big[x \leq y \to \exists t_1 \exists t_2\big(E_1(t_1,t_2) \land f(t_1) <$$
$$< x \leq y < f(t_2)\big)\big] \land \big[x > y \to \exists t_1 \exists t_2\big(E_1(t_1,t_2) \land$$
$$\land f(t_1) < y < x < f(t_2)\big)\big]$$

… … …

$$E'_{n-1}(x,y) \coloneqq \big[x \leq y \to \exists t_1 \exists t_2\big(E_{n-1}(t_1,t_2) \land$$
$$< f(t_2)\big)\big] \land \big[x > y \to \exists t_1 \exists t_2\big(E_{n-1}(t_1,t_2) \land$$
$$\land f(t_1) < y < x < f(t_2)\big)\big]$$

One can understand that $E'_1(x,y), \ldots,$ $E'_{n-1}(x,y)$ are equivalence relations partitioning $q(M)$ into infinitely many infinite convex classes and $E'_1(b,M) \subset \cdots \subset E'_{n-1}(b,M)$, whence we have $RC_{bin}(q) \geq n$ which contradicts our assumption.

(3) ($\Rightarrow$) (4). By Theorem 1.5 there exist only finitely many $\emptyset$-definable equivalence relations partitioning $q(M)$ into infinitely many infinite convex classes. Therefore, there exists a maximal $\emptyset$-definable equivalence relation with this property.

(4) ($\Rightarrow$) (1). Let for any (p,q)-splitting formula $R(x,y)$ the function $f^R(x) \coloneqq \sup R(x,M)$ is constant on each E-class. Clearly,

$$RC_{bin}(p) = RC_{bin}(q) + RC_{bin}(E(a, M)).$$
$$for\ some\ (any)\ a \in p(M)$$

Obviously, $E(x, y) \equiv E_i(x, y)$ for some $1 \le i \le n-1$. Then $RC_{bin}(E(a, M)) = i$ for any $a \in p(M)$. Fix an arbitrary (p, q)-splitting formula $R(x, y)$ and consider the behaviour of the function f^R on each $E_{i+1}(a, M)/E_i$, where $a \in p(M)$. The function f^R can not be constant on each $E_{i+1}(a, M)/E_i$, since otherwise f^R is constant on each E_{i+1}-class which contradicts maximality of $E_i(x, y)$ with this property. Consequently, f^R must be strictly monotonic on each $E_{i+1}(a, M)/E_i$, since otherwise if it is locally monotonic (non-strictly monotonic) on each $E_{i+1}(a, M)/E_i$, then an $\emptyset$-definable equivalence relation $E''(x, y)$ is appeared with the property that $E_i(a, M) \subset E''(a, M) \subset E_{i+1}(a, M)$ which contradicts the fact that E_{i+1} is an immediate successor of $E_i(x, y)$ among all $\emptyset$-definable equivalence relations on $p(M)$. Similarly, we can prove that the function f^R is strictly monotonic on each $E_{k+1}(a, M)/E_k$, where $i \le k \le n-2$ and f^R is strictly monotonic on $p(M)/E_{n-1}$.

Consider the following formulas:

$$E'_{i+1}(x, y) := \exists t_1 \exists t_2 [E_{i+1}(t_1, t_2) \wedge$$
$$\wedge f(t_1) < x < f(t_2) \wedge f(t_1) < y < f(t_2)]$$
$$\dots\ \dots\ \dots$$
$$E'_{n-1}(x, y) := \exists t_1 \exists t_2 [E_{n-1}(t_1, t_2) \wedge$$
$$\wedge f(t_1) < x < f(t_2) \wedge f(t_1) < y < f(t_2)]$$

Then it can be established that $E'_{i+1}(x, y), \dots, E'_{n-1}(x, y)$ are equivalence relations partitioning $q(M)$ into infinitely many infinite convex classes so that $E'_{i+1}(b, M) \subset \cdots \cdots \subset E'_{n-1}(b, M)$, whence $RC_{bin}(q) \ge n - i$.

Further, if there exists an $\emptyset$-definable equivalence relation $E^q(x, y)$ partitioning $q(M)$ into infinitely many infinite convex classes with the property $E^q(b, M) \subset E'_{i+1}(b, M)$, then consider the following formula:

$$E^*(x, y) := \exists t_1 \exists t_2 [E^q(t_1, t_2) \wedge$$
$$\wedge f(x) < t_1 < f(y) \wedge f(x) < t_2 < f(y)]$$

Obviously, $E_i(a, M) \subset E^*(a, M) \subset E_{i+1}(a, M)$ which also contradicts the fact that E_{i+1} is an immediate successor of E_i among all $\emptyset$-definable equivalence relations on $p(M)$. Similarly, we can prove that there is no $\emptyset$ -definable equivalence relation $E^q(x, y)$ partitioning $q(M)$ into infinitely many infinite convex classes so that $E'_k(b, M) \subset E^q(b, M) \subset \subset E'_{k+1}(b, M)$ for any k with $i + 1 \le k \le n-2$ or $E'_{n-1}(b, M) \subset E^q(b, M)$.

Thus, $RC_{bin}(q) = n - i$, i.e.,

$$RC_{bin}(p) = RC_{bin}(q) + RC_{bin}(E(a, M)).$$

Corollary 2.3. Let T be an almost omega-categorical weakly o-minimal theory, $p, q \in S_1(\emptyset)$ be non-algebraic, p is not weakly orthogonal to q. Then the following conditions are equivalent:

(1) $RC_{bin}(p) = RC_{bin}(q)$;

(2) there exists a (p, q)-splitting formula $R(x, y)$ such that $Range\ f^R{}_{p(M)}$ is everywhere dense in $q(M^{eq})$ and for any $\emptyset$-definable equivalence relation $E(x, y)$ partitioning $p(M)$ into infinitely many infinite convex classes the function f^R is not constant on each E-class;

(3) there exists a (p, q)-splitting formula $R(x, y)$ such that $Range\ f^R{}_{p(M)}$ is everywhere dense in $q(M^{eq})$ and the function f^R is locally monotonic (not locally constant) on $p(M)$.

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