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APPROXIMATIONS OF REGULAR GRAPHS

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Abstract. *The paper [11] raised the question of describing the cardinality and types of approximations for natural families of theories. In the present paper, a partial answer to this question is given, and the study of approximation and topological properties of natural classes of theories is also continued. We consider a cycle graph consisting of one cycle or, in other words, a certain number of vertices (at least 3 if the graph is simple) connected into a closed chain. It is shown that an infinite cycle graph is approximated by finite cycle graphs. Approximations of regular graphs by finite regular graphs are considered. On the other hand, approximations of acyclic regular graphs by finite regular graphs are considered. It is proved that any infinite regular graph is pseudofinite. And also, for any k, any k-regular graph is homogeneous and pseudofinite. Examples of pseudofinite 3-regular and 4-regular graphs are given.*

Key words: *regular graph, approximation of a theory, pseudofinite theory.*

ТҰРАҚТЫ ГРАФТАРДЫҢ АППРОКСИМАЦИЯЛАРЫ

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Андатпа. *[11] Жұмыста теориялардың табиғи үйірлері үшін аппроксимациялардың қуаты мен түрлерін сипаттау мәселесі көтерілген. Бұл жұмыста қойылған сұраққа ішінара жауап берілген және біз теориялардың табиғи класстарының аппроксимацияларын зерттеуді жалғастырамыз. Бір циклден немесе басқаша айтқанда, тұйық тізбекте қосылған шыңдардың белгілі бір санынан (граф қарапайым болса, кемінде 3) тұратын граф цикл қарастырылады. Шексіз граф цикл ақырлы граф циклдармен жуықталатыны көрсетілген. Тұрақты графтардың ақырлы тұрақты графтар арқылы аппроксимациялары қарастырылады. Сонымен қатар, ациклдік графтардың ақырлы тұрақты графтар арқылы аппроксимациялары қарастырылады. Шексіз тұрақты графтың псевдоақырлы екені дәлелденді. Сондай-ақ, кез келген k үшін кез келген k-тұрақты граф біртекті және псевдоақырлы екені дәлелденді. Псевдоақырлы 3-тұрақты және 4-тұрақты графтардың мысалдары келтірілген.*

Түйінді сөздер: *тұрақты граф, теориялар аппроксимациясы, псевдоақырлы теория.*

АППРОКСИМАЦИИ РЕГУЛЯРНЫХ ГРАФОВ

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Аннотация. *В работе [11] поставлен вопрос об описании мощности и видов аппроксимаций для естественных семейств теорий. В настоящей работе дается частичный ответ на этот вопрос, а также продолжается изучение аппроксимации и топологических свойств естественных классов теорий. Рассмотрен граф цикл, состоящий из одного цикла, или, другими словами, некоторого количества вершин (не менее 3, если граф простой), соединенных в замкнутую цепь. Показано, что бесконечный граф цикл аппроксимируется конечными графами циклами. Рассмотрены аппроксимации регулярных графов конечными регулярными графами. С другой стороны, рассмотрены аппроксимации ациклических регулярных графов конечными регулярными графами. Доказано, что любой бесконечный регулярный граф псевдоконечен. А также для любого k любой k-регулярный граф является однородным и псевдоконечным. Приведены примеры псевдоконечных 3-регулярных и 4-регулярных графов.*

Ключевые слова: *регулярный граф, аппроксимация теории, псевдоконечная теория.*

Introduction

A *graph* is an anlgebraic system $\Gamma = \langle G, R\rangle$, where R is a binary predicate symbol. The elements of the universe G are called the *vertices* of the graph Γ, and the elements of the binary relation $R \subseteq G^2$ are *arcs*. If (*a*,*b*) and (*b*,*a*) are arcs then the set $[a,b] = \{(a,b),(b,a)\}$ is called an *edge*. It is identified with the arcs (a,b) and (b,a). This edge u connects the vertices a and b, which are called the *endpoints* of u. If a vertex $a \in G$ is an endpoint of an edge $u \in R$, then a and u are *incident*. The *degree* of a vertex a in a graph Γ, written $deg_\Gamma(a)$ or simply $deg(a)$ is the number of edges incident to a, except that each loop at a counts twice. A vertex of degree 0 is called *isolated*, a vertex of degree 1 is called a *hanging* vertex. A graph that contains no cycles is called an *acyclic graph*. A connected acyclic graph is called a *tree*. Any graph without cycles is also called a forest so that the connected components of a forest are trees. Subsystems of the graph $\Gamma = \langle G, R\rangle$ are called *subgraphs*.

A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. If for two vertices $a, b \in G$ there is a path connecting them, then there is sure to be a minimal path connecting these vertices. We denote the length of this path by $\rho(a,b)$. If Γ has no such path, then $\rho(a,b) = \infty$. A tree is a path if and only if $deg(a) \leq 2$ for each vertex a of the tree.

Definition [12]. example For a tree fixed form vertex a, well the graph value $e(a) \triangleq max\{\rho(a,b): b \in G\}$ is called the *eccentricity* of a. The eccentricity of a vertex is equal to the distance from this vertex to the most distant from it. The maximum among all the eccentricities of the vertices is called the *diameter* of the graph Γ and is denoted by $d(\Gamma)$: $d(\Gamma) \triangleq max\{e(a): a \in G\}$. A vertex a is called *peripheral* if $e(a) = d(\Gamma)$. The minimal eccentricity of the graph Γ is called its *radius* and is denoted by $r(\Gamma)$: $r(\Gamma) \triangleq min\{e(a): a \in G\}$. The vertex a is called *central* if $e(a) = r(\Gamma)$. The set of all central vertices of a graph is called its *center*.

Definition [4]. An infinite graph $\Gamma = \langle G, R\rangle$ of the form $R = \{(a_0,a_1),(a_1,a_2),(a_2,a_3),\ldots\}$, $G = \{a_0, a_1, \ldots\}$ is called a *ray*, and a *double ray* is an infinite graph $\Gamma = \langle G, R\rangle$ of the form

$$G = \{\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\},$$

$$R = \{\ldots, (a_{-2}, a_{-1}), (a_{-1}, a_0), (a_0, a_1), (a_1, a_2), \ldots\};$$

in both cases the a_n's are assumed to be distinct.

Definition [7]. A *regular graph* is a graph where each vertex has the same number of neighbors. A regular graph with vertices of degree k is called a *k-regular graph* or *regular graph of degree* k.

Definition [6, 9]. A graph $\Gamma = \langle G, R \rangle$ is said to be *homogeneous* if, for $U, V \subseteq G$, the statement that $\langle U, R \upharpoonright U^2 \rangle \equiv \langle V, R \upharpoonright V^2 \rangle$ implies the existence of an automorphism of Γ mapping U to V.

In this paper, we consider a *cycle graph* consisting of one cycle or, in other words, a certain number of vertices (at least 3 if the graph is simple) connected into a closed chain. The cycle graph with n vertices is denoted by $C_n C_n$. Every vertex of C_n has degree 2. We will consider approximations of regular graphs.

In 1965 James Ax [1] investigated fields F having the property that *every absolutely irreducible variety over F has a F-rational point.* It was shown that the non-principal ultraproduct of finite fields has such property. Yuri Leonidovich Ershov in [5] called such fields *regularly closed*. In 1968, James Ax, in his work [2], first introduced the concept of *pseudofiniteness* to show the decidability of the theory of all finite fields, i.e. there is an algorithm to decide whether a given statement is true for all finite fields. It was proved that pseudofinite fields are exactly those infinite fields that have every elementary property common to all finite fields, that is, pseudofinite fields are infinite models of the theory of finite fields. He defined pseudofiniteness as follows:

Definition. A field F is *pseudofinite* if F is perfect, quasifinite and regularly closed.

The concept of "anotherpseudofinite structure" was first used in 1991 in the report of E. Hrushovski in meeting on Finite and Infinite Combinatorics in Sets and Logic [8], as well as in the joint works by E. Hrushovski and G. Cherlin. The following definition first occurs in [3]:

Definition. Let L be a language. An L-structure M is *pseudofinite* if for all L-sentences φ, $M \vDash \varphi$ implies that there is a finite M_0 M_0 such that $M_0 \vDash \varphi$. The elementary theory $T = Th(M)$ of a pseudofinite structure M is called *pseudofinite*.

Definition [11] Let $\mathcal{T}$ be a family of theories and T be a theory such that $T \notin T$. The theory T is said to be $\mathcal{T}$-*approximated*, or *approximated* by the family $\mathcal{T}$, or a pseudo-$\mathcal{T}$-theory, if for any formula $\varphi \in T$ there exists $T' \in T$ for which $\varphi \in T'$. If a theory T is $\mathcal{T}$-approximated, then $\mathcal{T}$ is said to be an *approximating family* for T, and theories $T' \in T$ are said to be approximations for T.

We put $T_\varphi = \{T \in T : \varphi \in T\}$. Such a set T_φ is called the *φ-neighbourhood*, or simply a *neighbourhood* for T. An approximating family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(T)$, T_φ is finite or $T_{\neg\varphi}$ is finite. It was shown in [7] that any e-minimal family $\mathcal{T}$ has a unique accumulation point T with respect to neighbourhoods T_φ, and $T \cup \{T\}$ is also called *e- minimal*.

Recall that the *E-closure* $Cl_E(T)$ [10] for the family T of complete theories is characterized by the following proposition.

Proposition 1. Let T be a family of complete theories of the language Σ. Then $Cl_E(T) = T$ for finite T and for infinite T, the theory T belongs to $Cl_E(T)$ if and only if T is a complete theory of the language Σ and T $\in T$, or $T \notin T$ and for of any formula $\varphi \in T$ the set T_φ is infinite.

We denote by $\acute{T}$ the class of all complete elementary theories, by $\acute{T}_{fin}$ the subclass of $\acute{T}$ consisting of all theories with finite models.

Proposition 2. [11] *For any theory T the following conditions are equivalent:*

(1) *T is pseudofinite;*

(2) *T is $\acute{T}_{fin}\overline{T}_{fin}$- approximated;*

(3) $T \in Cl_E(\acute{T}_{fin}) \setminus \acute{T}_{fin}$.

Main results

The following proposition shows that an infinite cycle graph is approximated by finite cycle graphs.

Proposition 3. Any theory T of a cycle graph on an infinite set is pseudofinite.

Proof. Let Γ be a model of the theory T and a be a vertex. For Γ, the following is true: $\Gamma = \lim_{i\to\infty} C_i$, where $C_i = C_{i-1} \cup \{a\}$ is finite, $i \geq 4$. That is, adding to C_i new vertices a of degree 2, in other words, increasing the distance between any pairs of vertices from C_i

in the limit, we obtain an infinite linear graph (or double ray), which is acyclic. The double ray Γ has no hanging vertices. Since all vertices have degree 2, there is an automorphism ϕ that maps any vertex a_i with $deg(a_i) = 2$ to a vertex a_j with $deg(a_j) = 2$ and $a_i \neq a_j$. Thus, $\{Th(C_{i-1} \cup \{a\}): i \in \omega\}$ approximates the theory $T = Th(\Gamma)$.

Theorem 1. Any theory T of a regular graph with an infinite set is pseudofinite.

Proof. We prove by induction on the degrees of vertices. definition Let $\Gamma = \langle G, R\rangle$ be a regular graph. Let m fbe the degree of vertices.

Let $m = 0$ or $m = 1$. Then, for the model regular Γ of the theory T, it is true that $\Gamma = \prod_{i\to\infty} \Gamma_i$, where Γ_i is a finite acyclic graph with a finite number of connected components, where each of them is a vertex of degree 0 or an edge. This means that by increasing the number of connected components step by step, we can construct a pseudofinite graph Γ.

The case $m = 2$ is considered in Proposition 2. Let $m = k$, where $k \geq 2$, and Γ_0' be the k- regular graph with $2(k-1)$ vertices. For a finite t, adding new $k(k-1)^t$ vertices at each step t, as a result we obtain a graph with $2(k-1) + \sum_{i=1}^{t} k(k-1)^t$ vertices of degree k. Continuing the process, in the limit, the graph is divided into acyclic connected components (trees). Since any infinite regular tree is vertex transitive, any route of length s s can be mapped to another s s-route. And this mapping can be extended to an automorphism of the acyclic regular graph Γ, which implies pseudofiniteness.

Then for $m = k + 1$ the graph Γ' is also pseudofinite. Similarly, taking a $(k+1)$-regular graph with vertices and adding $k^t(k+1)$ new vertices at each step t, in the limit we obtain an acyclic regular graph. Similarly, take an s-route and a vertex a_1 from this route that has $(k+1)$ neighbors, we map a_1 to another vertex a_2 of another s-route. The set of neighbors of the vertex a_1 can also be bijectively transferred to the set of neighbors of the vertex a_2.

Example 1. For clarity, as an example, we show the validity of the assertion for 2-regular and 3- regular, as well as 4-regular graphs. The pseudofiniteness of 2-regular graphs is proved in Proposition 2, and for $m = 3$ it is shown in Fig. 1. The tetrahedron Γ_0 is taken. At each step $i > 0$, adding vertices, in the limit we obtain an acyclic graph $\Gamma = i\Gamma_i$, where Γ_i is a finite regular graph.

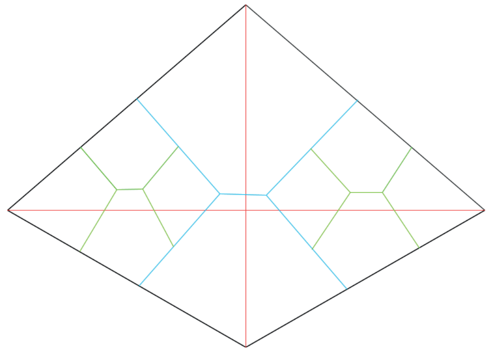

Figure 1 – Approximation of a 3-regular graph.

For any finite t, any 3-regular graph Γ consists of $4 + \sum_{i=1}^{t} 3 \cdot 2^i$ vertices. The infinite 3-regular graph Γ is split into acyclic components.

Example 2. In case $m = 4$, we take the octahedron Γ. Every i-th stage adding new vertices in the limit we get an acyclic graph (see Fig. 2). For a finite step r, the graph has $6+\sum_{i=1}^{r} 4 \cdot 3^i$ vertices of degree 4. Take any two routes of same length s as the induced subgraph and map one to another s-route, we can see that the mapping extends to an automorphism of the pseudofinite graph Γ.

From the above statement and examples it immediately follows:

Theorem 2. For any infinite regular graph Γ, the following conditions are true:

1. Γ is pseudofinite;
2. Γ is homogeneous.

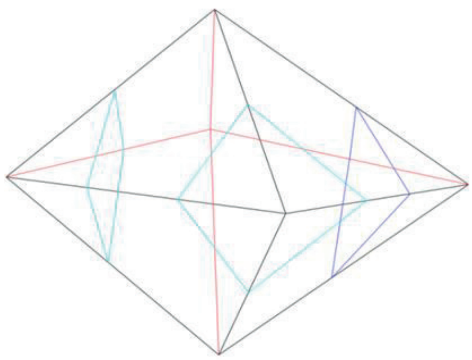

Figure 2 – Approximation of a 4-regular graph.

Conclusion

In samethis paper, we study approximations of regular graphs with finite ones. It is shown that the approximation in the limit gives an acyclic regular graph. It is proved that any theory T of regular graphs on an infinite set is pseudofinite. When approximating some graphs, there is a case when, in the limit, a graph with cycles is obtained. To get an acyclic graph, one can use Proposition 2 and break the cycles into two rays. For further study of various graph approximations, the following question can be posed:

Question: Which graphs defined by their automorphisms are pseudofinite?

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