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ON 1-INDISCERNIBILITY OF E-COMBINATIONS OF ORDERED THEORIES

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Abstract. *In this paper, we investigate properties that are preserved or acquired when combining an arbitrary number of theories or structures. Recently, an interest has been shown in the study of P-combinations (when each structure is distinguished by a separate unary predicate) and E-combinations (when each structure is distinguished by a separate class of equivalence with respect to E). Here we studied the properties of E-combinations of linearly ordered theories. The 1-indiscernibilty and density of a weakly o-minimal E-combination of countably many copies of an almost omega-categorical weakly o-minimal theory in a language that does not contain distinguished constants are established.*

Keywords: *linearly ordered structure, weak o-minimality, E-combination, omega-categoricity, dense ordering.*

РЕТТЕЛГЕН ТЕОРИЯЛАРДЫҢ Е-КОМБИНАЦИЯЛАРЫНЫҢ 1-АЖЫРАТЫЛМАУЫ ТУРАЛЫ

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Аңдатпа. *Бұл мақалада біз теориялардың немесе құрылымдардың еркін санын біріктіру кезінде сақталған немесе алынған қасиеттерді зерттейміз. Соңғы уақытта P-комбинацияларын (әрбір құрылым жеке унарлы предикатпен ерекшеленетін кезде) және E-комбинацияларын (әр құрылым E-ге қатысты эквиваленттіктің жеке класымен ерекшеленетін кезде) зерттеуге қызығушылық танытуда. Мұнда сызықтық реттелген теориялардың E-комбинацияларының қасиеттерін зерттедік. 1-айырықша константалары жоқ тілдегі дерлік омега-категориялық әлсіз o-минималды теорияның көшірмелерінің есептік санының әлсіз o-минималды E-комбинациясының 1-айырылмауы және тығыздығы белгіленді.*

Түйінді сөздер: *сызықтық реттелген құрылым, әлсіз o-минималдық, E-комбинациясы, омега-категориялық, тығыз рет.*

ОБ 1-НЕРАЗЛИЧИМОСТИ E-КОМБИНАЦИЙ УПОРЯДОЧЕННЫХ ТЕОРИЙ

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Аннотация. *В настоящей работе мы исследуем свойства, которые сохраняются или приобретаются при комбинировании произвольного числа теорий или структур. В последнее время интерес был проявлен к изучению P-комбинаций (когда каждая структура выделяется отдельным унарным предикатом) и E-комбинаций (когда каждая структура выделяется отдельным классом эквивалентности по отношению E). Здесь мы изучали свойства E-комбинаций линейно упорядоченных теорий. Установлены 1-неразличимость и плотность слабо o-минимальной E-комбинации счетного числа копий почти омега-категоричной слабо o-минимальной теории в языке, не содержащем выделенных констант.*

Ключевые слова: *линейно упорядоченная структура, слабая o-минимальность, E-комбинация, омега-категоричность, плотный порядок.*

Introduction

Earlier, in [1]–[12], various combinations of theories were considered. In this paper, we continue the study of combinations, namely, we will consider E-combinations of almost ω-categorical weakly o-minimal theories.

Let us introduce the necessary definitions. The notion of *weak o-minimality* was originally investigated by D. Macpherson, D. Marker, and C. Steinhorn in [13]. A subset A of a linearly ordered structure M is said to be *convex* if for any $a, b \in A$, and $c \in M$, whenever $a < c < b$, we have $c \in A$. A *weakly o-minimal structure* is a linearly ordered structure $M = \langle M, <, \ldots \rangle$ such that any definable (with parameters) subset of the structure M is the union of finitely many convex sets in M.

Definition 1. [14, 15] Let T be a complete theory, $p_1(x_1), \ldots, p_n(x_n) \in S_1(\emptyset)$. An n-type $q(x_1, \ldots, x_n) \in S_n(\emptyset)$ is said to be a $(p_1, \ldots, p_n)$*-type* if $q(x_1, \ldots, x_n) \supseteq \bigcup_{i=1}^{n} p_i(x_i)$. The set of all $(p_1, \ldots, p_n)$-types of the theory T is denoted by $S_{p_1,\ldots,p_n}(T)$. A countable theory T is said to be *almost ω-categorical* if for any types $p_1(x_1), \ldots, p_n(x_n) \in S_1(\emptyset)$ there exist only finitely many types $q(x_1, \ldots, x_n) \in S_{p_1,\ldots,p_n}(T)$.

Almost ω-categoricity is closely connected with the notion of Ehrenfeuchtness of a theory. Thus, in the work [14] it was proved that if T is an almost ω-categorical theory with $I(T, \omega) = 3$, then a dense linear ordering is interpreted in the theory T.

Throughout this article, we will consider linearly ordered structures, i.e. structures of a language containing a binary relation symbol $<$ that satisfies the axioms of a linear order.

Let M_i be a linearly ordered structure of the signature $\{<, \Sigma_i\}$ for each $i < \omega$, where Σ_i does not contain distinguished constants. We will denote by $dcl^{<}_{M_i}(\emptyset)$ the sets of elements of a structure M_i being $\emptyset$-definable by the order realtion $<_{M_i}$.

We will say that $M^+ := \langle \bigcup_{i\in\omega} M_i; <, \Sigma, E^2, c^i_k \rangle_{k<\lambda_i, i\in\omega}$ is a *linearly ordered disjoint E-combination* (or just *E-combination*) of the structures M_i, if $\Sigma := \bigcup_{i\in\omega} \Sigma_i$, $\{c^i_k \mid k < \lambda_i\} \subseteq dcl^{<}_{M_i}(\emptyset)$ for some ordinal λ_i; either $M_l < M_m$ or $M_m < M_l$ for any $l, m \in \omega$, and E is an equivalence relation partitioning M^+ into convex classes so that for any $a \in M^+$ we have $E(a, M^+) = M_i$ for some $i < \omega$, $E \notin \Sigma$.

Thus, we include in the sigmature of an arbitrary E-combination of structures M_i, $i \in \omega$, all elements lying in $dcl^{<}_{M_i}(\emptyset)$ for every $i \in \omega$, i.e. if M_1 and M_2 are isomorphic copies of the same structure M, that has λ elements lying in $dcl^{<}_{M}(\emptyset)$ for some cardinal λ, then in the signature of an E-combination of the structures M_1 and M_2 exactly 2λ elements will be included.

Here we are interested in the questions of preserving certain properties of the original structures in their E-combination. For example, if all M_i are almost ω-categorical, then under what conditions will an elementary theory of an arbitrary E-combination of these structures be almost ω-categorical as well? Or when will it have the maximal countable spectrum?

Results

Fact 1. Let T_i be an almost ω-categorical weakly o-minimal theory for every $i \in \omega$, $M_i \models T_i$, M^+ be a linearly ordered disjoint E-combination of finitely many such models. Then $Th(M^+)$ is an almost ω-categorical weakly o-minimal theory.

We say that a tuple $\bar{a} := \langle a_1, \dots, a_n \rangle \in M^n$ forms a *finite linear ordering* or $F(n)$*-ordering* if $a_1 < a_2 < \dots < a_n$, a_1 does not have an immediate predecessor in M, a_n does not have an immediate successor in M, and a_{i+1} is an immediate successor of a_i for every $1 \le i \le n-1$.

Example 1. Let $M := \langle \mathbb{Q}, <, P_i^1 \rangle_{i\in\omega}$ be a linearly ordered structure, $\mathbb{Q}$ be the set of rational numbers, $P_i(M) = \{b \in \mathbb{Q} | b < \sqrt{2} + i\}$ for every $i \in \omega$. Then, obviously, $P_i(M)$ is convex for every $i \in \omega$ and

$$P_0(M) \subset P_1(M) \subset P_2(M) \subset \dots \subset P_n(M) \subset \dots$$

Observe that since the structure $M_0 := \langle \mathbb{Q}, < \rangle$ is o-minimal then by Theorem 63 [16] $Th(M)$ is weakly o-minimal. Consider the following set of formulas:

$$\{\forall y[P_i(y) \to y < x] \mid i \in \omega\}$$

It is locally consistent and determines a complete type over $\emptyset$. Denote it by $p(x)$. This type is non-isolated; the set of realizations of p can be empty, have the ordering type $[0,1) \cap \mathbb{Q}$ or $(0,1) \cap \mathbb{Q}$. Thus, $Th(M)$ has exactly three countable pairwise non-isomorphic models, i.e. is Ehrenfeucht. Consequently, by Theorem 3.7 [17] $Th(M)$ is almost ω-categorical.

Let M^+ be a linearly ordered disjoint E-combination of countably many copies of the model M. Obviously, M^+ is not weakly o-minimal, since $P_0(M^+)$ is an union of infinitely many convex sets.

We assert that independently from that how E-classes are ordered in M^+, the theory $Th(M^+)$ has 2^ω countable models.

Case 1. E-classes are densely ordered without endpoints.

Let

$$p^+(x) := \{\forall y[P_i(y) \to y < x \wedge E(y,x)] \mid i \in \omega\}$$

This set of formulas ia consistent and determines a complete type over $\emptyset$ in $Th(M^+)$. The type p^+ can be omitted in every concrete class of equivalence, i.e. the set of realizations of p can be empty, can have the ordering type $[0,1) \cap \mathbb{Q}$ or $(0,1) \cap \mathbb{Q}$. Let's select an arbitrary ω equivalence classes with the least left class: E_0, E_1, E_2, ..., i.e. there exist $a_0, a_1, a_2, \dots$ such that

$$E_0 = E(a_0, M^+), E_1 = E(a_1, M^+), E_2 = E(a_2, M^+), \dots$$
$$\text{and } E_0 < E_1 < E_2 < \cdots$$

We will consider only countable models of the theory $Th(M^+)$, in which $p^+(E_i) \neq \emptyset$ for any $i \in \omega$, and in the remaining equivalence classes the set of realizations of type p^+ is empty. We define the following encoding: if one of these classes is implemented by the set of realizations with the smallest element, then we encode it by 1, but if it is implemented by the set of realizations without the smallest element, then we encode it by 2. Since the set of all possible countable sequences from 1 and 2 is of cardinality continuum, we conclude that $Th(M^+)$ has 2^ω countable models.

We also assert that in this case $Th(M^+)$ is almost ω-categorical.

Case 2. E-classes are ordered by the type ω.

Consider the following formulas:

$$\phi_1(x) := \forall y[\neg E(x,y) \to x < y],$$
$$\phi_n(x) := \forall y[y < x \wedge \neg E(x,y) \to \vee_{i=1}^{n-1} \phi_i(y)], n \ge 2.$$

Obviously, $\phi_1(x)$ defines the leftmost equivalence class, $\phi_2(x)$ defines the second class, $\phi_n(x)$ definces the n-th equivalence class for each $n < \omega$.

Consider the following set of formulas:

$$p_0(x) := \{\forall y[\phi_n(y) \to y < x] \mid n \in \omega\}.$$

It is locally consistent. Consequently, there exists $M_1^+ \succ M^+$, in which $p_0(x)$ is realized by countably many E-classes ordered by the type $\omega + \omega^*$.

Consider the following formulas:

$$s_{1,E}(x,y) := x < y \wedge \neg E(x,y) \wedge$$

$$\wedge \forall z(x \leq z \leq y \to E(x,z) \vee E(z,y)),$$

$$s_{n,E}(x,y) := x < y \wedge \neg E(x,y) \wedge$$

$$\exists t_1 \dots \exists t_{n-1}[\neg E(x,t_1) \wedge \bigwedge_{i=1}^{n-2} \neg E(t_i,t_{i+1}) \wedge \neg E(t_{n-1},y)$$

$$\wedge x < t_1 < \dots < t_{n-1} < y \wedge \forall t(x \leq t \leq y \to$$

$$(E(x,t) \vee \bigvee_{i=1}^{n-1} E(t,t_i) \vee E(t,y))], n \geq 2.$$

Let $p(x) := p_0(x) \cup \{P_0(x)\}$. This set of formulas determines a complete type over $\emptyset$. Then considering for each natural $k \geq 1$ the following set of formulas:

$p(x) \cup p(y) \cup \{s_{k,E}(x,y)\}$,

we obtain that the number of (p_1, p_2)-types is infinite, where $p_i(x) := p(x)$, $i = 1,2$, and consequently $Th(M^+)$ is not almost ω-categorical.

Let $\mathbb{ZE}$ denote the set of E-classes orderd by the type $\omega^* + \omega$. Then we denote by $F(k)^{\mathbb{ZE}}$ ($\omega^{\mathbb{ZE}}$ and $\mathbb{Q}^{\mathbb{ZE}}$) the set of $\mathbb{ZE}$-copies ordered by the type $F(k)$ (ω and $\mathbb{Q}$ respectively). Then we assert that $p_0(x)$ can be realized by the following set:

$$F_1(k_1)^{\mathbb{ZE}} + \mathbb{Q}^{\mathbb{ZE}} + F_2(k_2)^{\mathbb{ZE}} + \mathbb{Q}^{\mathbb{ZE}} + \dots + F_n(k_n)^{\mathbb{ZE}} + \mathbb{Q}^{\mathbb{ZE}}$$

for any $0 \leq n, k_i \leq \omega$, where for every $2 \leq i \leq n-1$ if $k_i \neq 0$ then $k_i \geq 2$; and if $k_i = \omega$ then $F_i(k_i)^{\mathbb{ZE}} \equiv \omega^{\mathbb{ZE}}$, whence we obtain that $Th(M^+)$ has 2^ω countable models.

Theorem 1. Let T be an almost ω-categorical weakly o-minimal theory of a language non-containing distinguished constants, $\mathrm{M} \models \mathrm{T}$, M^+ be a linearly ordered disjoint E-combination of countably many copies of the structure M. Suppose that $\mathrm{Th}(\mathrm{M}^+)$ is weakly o-minimal. Then the following holds:

(1) M^+/E is partitioned into finitely many convex sets on each of which either all the elements have both an immediate predecessor and an immediate successor or all the elements have neither an immediate predecessor not an immediate successor.

(2) M is dense.

(3) M is 1-indiscernible.

Proof of theorem 1. Consider the following formula:

$$\theta(x) := \exists t \exists y[t < x < y \wedge \neg E(t,x) \wedge \neg E(x,y) \wedge$$

$$\forall u \forall z(t < u < x < z < y \to$$

$$(E(t,u) \vee E(u,x)) \wedge (E(x,z) \vee E(z,y))].$$

Since $Th(M^+)$ is weakly o-minimal, the set $\theta(M^+)$ is an union of finitely many convex sets, and therefore (1) holds.

Prove now that M is dense. If M is not dense then there exist elements in M having an immediate predecessor or an immediate successor. By both almost ω-categoricity and weak o-minimality of the theory T there exist only finitely many elements in M having an immediate predecessor or an immediate successor. Consider the following formula:

$$\phi(x) := \exists y[x < y \wedge \forall z(x < z \to y \leq z)].$$

Obviously, $\phi(M^+)$ is a union of infinitely many $\neg\phi(M^+)$-separable convex sets, whence M^+ is not weakly o-minimal that contradicts the hypotheses of the theorem.

We prove now that M is 1-indiscernible. If this is not true then there exist $a, b \in M$ such that $a \neq b$ and $tp(a/\emptyset) \neq tp(b/\emptyset)$. Consequently, there exists an L-formula $\psi(x)$ such that

$$M| = \psi(a) \wedge \neg\psi(b),$$

i.e. $\psi(M) \neq M$. By weak o-minimality we can assume that $\psi(M)$ is convex. But then $\psi(M^+)$ is an union of infinitely many $\neg\psi(M^+)$ -separable convex sets, that contradicts the weak o-minimality of M^+.

Corollary 1. Let T_i be an almost ω-categorical weakly o-minimal theory of a language non-containing distinguished constants for each $i \in$ ω, $M_i \vDash T_i$, M^+ be a linearly ordered disjoint E-combination of the structures M_i. Suppose that $Th(M^+)$ is weakly o-minimal. Then the following holds:

(1) M^+/E is partitioned into finitely many convex sets on each of which either all the elements have both an immediate predecessor and an immediate successor or all the elements have neither an immediate predecessor not an immediate successor.

(2) M_i is dense for almost all $i \in \omega$.

(3) M_i is 1-indiscernibel for almost all $i \in \omega$.

Conclusion

Examples of E-combinations of countably many almost omega-categorical weakly o-minimal theories were constructed, in which both weak o-minimality and almost omega-categoricity are not preserved. It also was established that the countable spectrum might change. At the same time, it has been proved that when considering an E-combination of countably many copies of an almost omega-categorical weakly o-minimal theory in a language without distinguished constants, if the resulting combination is weakly o-minimal then the initial structure should be dense and 1-indiscernible.

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