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ANALYTICAL AND APPROXIMATE SOLUTIONS OF THE TWO-DIMENSIONAL LAPLACE EQUATION IN CYLINDRICAL COORDINATES

Abstract

This paper investigates a steady-state heat conduction problem in a multilayer cylindrical domain with nonhomogeneous boundary conditions. Two approaches for solving the direct problem are considered: an analytical method based on separation of variables and Fourier series expansion, and a numerical method based on the finite difference approximation combined with the Successive Over-Relaxation (SOR) algorithm. An exact analytical solution of the Laplace equation in polar coordinates is derived for a three-layer cylindrical system with continuity conditions at layer interfaces and convective heat exchange at the outer boundary. A finite difference scheme is developed to approximate the governing equation, while the resulting system of linear algebraic equations is solved iteratively using the SOR method.The accuracy of the numerical approach is evaluated through comparison with the analytical solution and experimental temperature measurements. The obtained results demonstrate good agreement between the analytical and numerical solutions. The mean absolute error between the two approaches is $0.134K$, the root mean square error is $0.141K$, and the maximum absolute error does not exceed $0.209K$. Experimental validation shows that both models correctly reproduce the overall temperature distribution in the multilayer cylindrical system. The mean absolute errors relative to experimental data are $2.509K$ for the analytical solution and $2.638K$ for the SOR solution. The performed analysis confirms the good accuracy and stability of the developed finite difference algorithm and demonstrates its applicability for modeling heat transfer processes in multilayer cylindrical structures. The proposed approach can be employed in the thermal analysis of pipelines, geothermal systems, insulation structures, and other engineering applications involving layered media.

Keywords: Heat conduction; Laplace equation; multilayer cylindrical domain; analytical solution; finite difference method; Successive Over-Relaxation (SOR); temperature distribution; experimental validation; polar coordinates; thermal modeling.

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Introduction

Heat transfer in multilayer cylindrical structures is an important topic in both engineering and applied science due to its wide range of practical applications. Such structures are commonly found in oil and gas pipelines, geothermal systems, thermal insulation technologies, and various aerospace and nuclear engineering components [1]. The analysis of heat transfer in these systems is complicated by the presence of different materials, multiple interfaces between layers, and complex boundary conditions that often vary along the surface.

A key step in understanding the thermal behavior of multilayer systems is the solution of the direct heat conduction problem, which aims to determine the temperature distribution within the domain when the material properties and boundary conditions are known [2]. To address this problem, researchers typically employ either analytical methods, which can provide exact solutions under certain assumptions, or approximate approaches based on numerical techniques. Each of these methods offers specific advantages and faces particular limitations depending on the complexity of the physical model.

Heat transfer in multilayer cylindrical structures is encountered in many engineering applications, including oil and gas pipelines, geothermal wells, underground thermal energy storage systems, and multilayer insulation technologies. In such systems, the temperature field is affected not only by the thermal properties of individual materials but also by the interaction between layers and the conditions imposed on the boundaries. Reliable prediction of temperature distributions is therefore essential for evaluating thermal efficiency, structural integrity, and long-term performance.

For steady-state problems, analytical methods remain an attractive tool because they provide explicit relationships between temperature distributions and the governing parameters of the system. Such solutions are particularly useful when studying the influence of material properties, layer thicknesses, or boundary conditions. Yang and Liu [1] proposed an analytical transfer-function framework for multilayer cylinders with an arbitrary number of layers, while Kayhani et al. [2] derived closed-form solutions for composite cylindrical laminates composed of orthotropic materials. These approaches provide accurate results and valuable physical insight. However, their application becomes increasingly difficult when the geometry, material configuration, or boundary conditions become more complex.

A comprehensive review of analytical solutions for multilayer structures can be found in [3]. One of the main challenges in analytical modeling is the treatment of non-uniform boundary conditions. In cylindrical geometries, boundary temperatures and heat fluxes often vary along the angular coordinate because of external environmental effects, localized heating sources, or asymmetric operating conditions. Although analytical techniques based on Fourier expansions can handle such situations, the resulting formulations may become cumbersome, especially for multilayer domains. Additional difficulties arise in transient problems, where eigenvalue calculations and series representations may require considerable computational effort and can suffer from numerical instability [4].

Because of these limitations, numerical methods are widely used in practical thermal analysis. Among them, the finite difference method remains one of the most accessible and computationally efficient approaches. It can easily accommodate multilayer structures, heterogeneous materials, and complicated boundary conditions without requiring the derivation of lengthy analytical expressions. The price for this flexibility is the introduction of discretization errors, whose magnitude depends on the mesh resolution and numerical scheme. For this reason, the accuracy of a numerical solution should be carefully evaluated whenever a reliable analytical solution is available.

A detailed overview of analytical solutions for multilayer media was presented by Amiri Delouei et al. [6], who examined rectangular, cylindrical, and spherical geometries. Their review highlights how interface conditions, thermal contact resistance, internal heat generation, and boundary conditions influence the complexity of the resulting mathematical models. At the same time, the review shows that relatively few studies directly compare analytical and numerical solutions for the same multilayer configuration under identical conditions.

Several recent studies have focused on multilayer cylindrical systems. In [7], an analytical solution of the steady-state heat conduction equation in polar coordinates was developed for multilayer soils and validated through numerical examples. Later, the same analytical framework was employed in inverse heat conduction problems, where unknown material properties were reconstructed from temperature measurements [8]. More recently, experimental data were incorporated into the inverse analysis of multilayer cylindrical systems, enabling the estimation of thermophysical parameters from measured temperatures [9].

Despite these developments, the question of how accurately a finite difference model reproduces the corresponding analytical solution for the same multilayer cylindrical domain has received relatively little attention. Such a comparison is important not only for validating numerical algorithms but also for understanding where numerical errors arise and how they propagate in regions with strong temperature gradients or discontinuous material properties.

The present study addresses this issue for a multilayer cylindrical domain with non-uniform boundary conditions. An analytical solution is derived using separation of variables and Fourier series expansion, while the numerical solution is obtained using the finite difference method combined with the Successive Over-Relaxation (SOR) algorithm. The performance of the numerical approach is assessed through direct comparison with the analytical solution and experimental temperature measurements.

This study combines analytical, numerical, and experimental investigations of heat transfer in a multilayer cylindrical system. By comparing the analytical solution, the SOR-based numerical results, and experimental measurements, it is possible to evaluate the accuracy of each approach and better understand their strengths and limitations.

The remainder of this paper is organized as follows. Section presents the mathematical formulation of the problem, it derives the analytical solution, and describes the numerical algorithm based on the finite difference method and the SOR technique. Section provides a comparative analysis of the analytical, numerical, and experimental results. Finally, the main conclusions of the study are summarized in Section .

Materials and methods

Mathematical Formulation of the Problem

Consider a steady-state heat conduction process in a multilayer cylindrical domain described in the two-dimensional polar coordinate system (r, φ), where r denotes the radial coordinate and φ is the angular coordinate. It is assumed that no internal heat sources are present within the domain and that the temperature field has reached a steady state. Under these assumptions, the temperature distribution in each layer $(s = 1, 2, 3)$ satisfies the Laplace equation in polar coordinates [10, 11]:

$$\frac{\partial^2 \tilde{T}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{T}}{\partial \tilde{r}} + \frac{1}{\tilde{r}^2}\frac{\partial^2 \tilde{T}}{\partial \varphi^2} = 0, \quad \tilde{r}_0 < \tilde{r} < \tilde{r}_3,\ 0 < \varphi < 2\pi. \tag{1}$$

A Dirichlet boundary condition is specified at the inner boundary of the cylinder. For the purposes of this study, the temperature at the inner boundary is assumed to be constant [12]:

$$\tilde{T}^{(1)}(\tilde{r}_0, \varphi) = \tilde{f}, \tag{2}$$

where $\tilde{f}$ is known.

At the layer interfaces, continuity of temperature and heat flux is required [13]:

$$\tilde{T}^{(s)}(\tilde{r}_s, \varphi) = \tilde{T}^{(s+1)}(\tilde{r}_s, \varphi), \tag{3}$$

$$k_s \frac{\partial \tilde{T}^{(s)}}{\partial \tilde{r}}(\tilde{r}_s, \varphi) = k_{s+1} \frac{\partial \tilde{T}^{(s+1)}}{\partial \tilde{r}}(\tilde{r}_s, \varphi), \quad s = 1, 2, \tag{4}$$

where k_s denotes the thermal conductivity of the s-th layer.

At the outer boundary, convective heat transfer with the surrounding environment is described by a Robin boundary condition [14, 15]:

$$\left.\frac{\partial \tilde{T}}{\partial \tilde{r}}\right|_{\tilde{r}=\tilde{r}_3} = -\frac{h}{k_3}\big(\tilde{T}(\tilde{r}_3, \varphi) - \tilde{T}_a(\varphi)\big), \tag{5}$$

where h is the convective heat transfer coefficient at the outer boundary of the third layer, and $\tilde{T}_a(\varphi)$ represents the ambient temperature distribution.

Furthermore, periodicity of the temperature field with respect to the angular coordinate is assumed:

$$\tilde{T}(\tilde{r}, \varphi) = \tilde{T}(\tilde{r}, \varphi + 2\pi). \tag{6}$$

The resulting mathematical model forms the foundation for both the analytical and numerical treatment of the heat conduction problem in a multilayer cylindrical domain.

To facilitate the analysis and subsequent numerical solution, dimensionless variables are introduced. The radial coordinate is normalized by the outer radius of the cylinder:

$$r = \frac{\tilde{r}}{\tilde{r}_3}, \quad \frac{\tilde{r}_0}{\tilde{r}_3} < r \leq 1.$$

The temperature is represented in dimensionless form (T) using a reference temperature chosen for normalization:

$$T = \frac{\tilde{T} - T_{\mathrm{ref}}}{T_{\mathrm{ref}}}, \quad T_{\mathrm{ref}} = 277K. \tag{7}$$

Here, T_{ref} is a fixed reference temperature introduced to scale the temperature field and to obtain a dimensionless representation [16, 17].

In dimensionless variables, the heat conduction equation retains the form of the Laplace equation:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} = 0, \tag{8}$$

while the Robin boundary condition at the external boundary is written in terms of the dimensionless Biot number [18]:

$$\left.\frac{\partial T}{\partial r}\right|_{r=1} = -\mathrm{Bi}\,\big(T(1, \varphi) - T_a(\varphi)\big), \quad \mathrm{Bi} = \frac{hr_3}{k_3}, \quad T_a(\varphi) = \frac{\tilde{T}_a(\varphi) - T_{\mathrm{ref}}}{T_{\mathrm{ref}}}. \tag{9}$$

The remaining boundary conditions (3)–(5) remain unchanged in form after the nondimensionalization. Thus, the dimensionless formulation of the problem is governed by the geometric parameters r_0, r_1, r_2, the thermal conductivities of the layers k_1, k_2, k_3 and the Biot number Bi.

Exact Solution of the Laplace Equation

The analytical solution is constructed using the method of separation of variables. Accordingly, the temperature field is sought in the form of a product of radial and angular functions [19]:

$$T(r,\varphi) = R(r)\,\Phi(\varphi). \tag{10}$$

Substitution of the separated representation into the Laplace equation leads to two independent ordinary differential equations for the angular and radial components. Owing to the periodicity condition in the angular coordinate, the solution of the angular equation takes the form:

$$\Phi_n(\varphi) = \tilde{A}_n \cos(n\varphi) + \tilde{B}_n \sin(n\varphi), \quad n = 0, 1, 2, \dots \tag{11}$$

For the harmonic modes $n \geq 1$ the radial solution takes the form:

$$R_n(r) = \overline{A}_n r^n + \overline{B}_n r^{-n}. \tag{12}$$

For the zeroth harmonic ($n = 0$), the radial solution is given by:

$$R_0(r) = \overline{A}_0 \ln r + \overline{B}_0. \tag{13}$$

Therefore, combining the radial and angular solutions yields the following Fourier series representation of the temperature field in each layer of the cylindrical domain:

$$T^{(s)}(r,\varphi) = A_0^{(s)} \ln r + B_0^{(s)} + \sum_{n=1}^{\infty} \left(A_n^{(s)} r^n + A_{-n}^{(s)} r^{-n} \right) \cos(n\varphi) + \sum_{n=1}^{\infty} \left(B_n^{(s)} r^n + B_{-n}^{(s)} r^{-n} \right) \sin(n\varphi). \tag{14}$$

where s is number of layer and $s = 1, 2, 3$, all unknown coefficients $A_0^{(s)}, B_0^{(s)}, A_n^{(s)}, A_{-n}^{(s)}, B_n^{(s)}, B_{-n}^{(s)}$ are obtained by enforcing the boundary conditions (2) and (5) as well as the continuity conditions for temperature and heat flux at the layer interfaces (3), (4):

$$A_0^{(2)} = \frac{1}{2\pi \cdot R} \left(\int_0^{2\pi} T_a(\varphi) d\varphi - \int_0^{2\pi} f d\varphi \right). \tag{15}$$

$$A_0^{(1)} = \frac{k_2}{k_1} A_0^{(2)}. \tag{16}$$

$$A_0^{(3)} = \frac{k_2}{k_3} A_0^{(2)}. \tag{17}$$

$$B_0^{(1)} = \frac{1}{2\pi} \int_0^{2\pi} f d\varphi - A_0^{(1)} \ln r_0. \tag{18}$$

$$B_0^{(3)} = \frac{1}{2\pi} \int_0^{2\pi} T_a(\varphi) d\varphi - \frac{A_0^{(3)}}{Bi}. \tag{19}$$

$$B_0^{(2)} = \left(\frac{k_2}{k_1} \big(\ln r_1 - \ln r_0 \big) - \ln r_1 \right) \cdot A_0^{(2)} + \frac{1}{2\pi} \int_0^{2\pi} f d\varphi. \tag{20}$$

$$A_{-n}^{(2)} = \frac{1}{Q \cdot M + S} \left(\frac{Bi}{\pi} \int_0^{2\pi} T_a(\varphi) cosn\varphi d\varphi - Q \cdot K \cdot \int_0^{2\pi} f \cdot \cos n\varphi d\varphi \right). \tag{21}$$

$$A_n^{(2)} = K \cdot \int_0^{2\pi} f \cdot \cos n\varphi d\varphi + M \cdot A_{-n}^{(2)}. \tag{22}$$

$$\begin{cases} A_n^{(1)} = \frac{k_1+k_2}{2k_1} A_n^{(2)} + \frac{k_1-k_2}{2k_1} r_1^{-2n} A_{-n}^{(2)}, \\ A_{-n}^{(1)} = \frac{k_2-k_1}{2k_1} r_1^{2n} A_n^{(2)} + \frac{k_1+k_2}{2k_1} A_{-n}^{(2)}. \end{cases} \tag{23}$$

$$\begin{cases} A_n^{(3)} = \frac{k_2+k_3}{2k_3} A_n^{(2)} - \frac{\left(k_2-k_3\right) r_2^{-2n}}{2k_3} A_{-n}^{(2)}, \\ A_{-n}^{(3)} = -\frac{k_2-k_3}{2k_3} r_2^{2n} A_n^{(2)} + \frac{k_2+k_3}{2k_3} A_{-n}^{(2)}. \end{cases} \tag{24}$$

$$B_n^{(2)} = \frac{1}{H - I \cdot U} \left(\frac{Bi}{\pi} \int_0^{2\pi} T_a(\varphi) sinn\varphi d\varphi - I \cdot O \cdot \int_0^{2\pi} f \cdot \sin n\varphi d\varphi \right). \tag{25}$$

$$B_{-n}^{(2)} = O \cdot \int_0^{2\pi} f \cdot \sin n\varphi d\varphi - U \cdot B_n^{(2)}. \tag{26}$$

$$\begin{cases} B_n^{(1)} = \frac{k_1+k_2}{2k_1} B_n^{(2)} + \frac{k_1-k_2}{2k_1} r_1^{-2n} B_{-n}^{(2)}, \\ B_{-n}^{(1)} = \frac{k_1-k_2}{2k_1} r_1^{2n} B_n^{(2)} + \frac{k_1+k_2}{2k_1} B_{-n}^{(2)}. \end{cases} \tag{27}$$

$$\begin{cases} B_n^{(3)} = \frac{k_2+k_3}{2k_3} B_n^{(2)} + \frac{k_3-k_2}{2k_3} r_2^{-2n} B_{-n}^{(2)}, \\ B_{-n}^{(3)} = \frac{k_3-k_2}{2k_3} r_2^{2n} B_n^{(2)} + \frac{k_2+k_3}{2k_3} B_{-n}^{(2)}. \end{cases} \tag{28}$$

All auxiliary coefficients $R, Q, M, S, K, H, U, I, O$ required in expressions (15)–(28) are explicitly derived as:

1. $R = \left(1 - \frac{k_2}{k_3}\right) lnr_2 + \left(\frac{k_2}{k_1} - 1\right) lnr_1 - \frac{k_2}{k_1} lnr_0 + \frac{1}{Bi} \cdot \frac{k_2}{k_3}$;
2. $Q = \frac{(n+Bi)(k_2+k_3)}{2k_3} + \frac{(n-Bi)(k_2-k_3)r_2^{2n}}{2k_3}$;
3. $M = -\frac{r_0^n r_1^{-2n}(k_1-k_2)+r_0^{-n}(k_1+k_2)}{r_0^n(k_1+k_2)+r_0^{-n} r_1^{2n}(k_1-k_2)}$;
4. $S = -\frac{(n+Bi)(k_2-k_3)r_2^{-2n}}{2k_3} - \frac{(n-Bi)(k_2+k_3)}{2k_3}$;
5. $K = \frac{2k_1}{\pi\left(r_0^n(k_1+k_2)+r_0^{-n} r_1^{2n}(k_1-k_2)\right)}$;
6. $H = \frac{(n+Bi)(k_2+k_3)}{2k_3} - \frac{r_2^{2n}(n-Bi)(k_3-k_2)}{2k_3}$;
7. $U = \frac{r_0^n(k_1+k_2)+r_0^{-n} r_1^{2n}(k_1-k_2)}{r_0^n r_1^{-2n}(k_1-k_2)+r_0^{-n}(k_1+k_2)}$;
8. $I = \frac{(n+Bi)r_2^{-2n}(k_3-k_2)}{2k_3} - \frac{(n-Bi)(k_2+k_3)}{2k_3}$;
9. $O = \frac{2k_1}{\pi\left(r_0^n r_1^{-2n}(k_1-k_2)+r_0^{-n}(k_1+k_2)\right)}$.

The zeroth harmonic describes the average component of the temperature field, while the harmonics with $n \geq 1$ capture its angular variations. By enforcing the boundary and interface conditions, a system of linear algebraic equations is obtained for the unknown expansion coefficients.

Consequently, the exact solution is expressed analytically as a Fourier series in the angular coordinate.

Approximate Solution Method

The numerical solution is obtained using the finite difference method combined with the Successive Over-Relaxation (SOR) algorithm. The computational domain is discretized in both the radial and angular coordinates. Because the cylindrical domain is composed of three layers, individual radial grid spacings are defined for each layer:

$$\Delta r_1 = \frac{r_1 - r_0}{N_1}, \qquad \Delta r_2 = \frac{r_2 - r_1}{N_2}, \qquad \Delta r_3 = \frac{r_3 - r_2}{N_3},$$

The discretization step in the angular coordinate is given by:

$$\Delta\varphi = \frac{2\pi}{M},$$

where N_1, N_2, N_3 are the number of grid points in each layer, and M — denotes the number of grid points in the angular direction.

After discretizing the Laplace equation (1) using central finite differences, the following finite difference equation is obtained for an interior grid point (r_i, φ_j):

$$A_i T_{i-1,j} + B_i T_{i+1,j} + C_i T_{i,j-1} + D_i T_{i,j+1} - E_i T_{i,j} = 0,$$

where the coefficients are defined as follows:

$$A_i = \frac{r_i - \frac{\Delta r_s}{2}}{r_i(\Delta r_s)^2},$$

$$B_i = \frac{r_i + \frac{\Delta r_s}{2}}{r_i(\Delta r_s)^2},$$

$$C_i = D_i = \frac{1}{r_i^2(\Delta\varphi)^2},$$

$$E_i = A_i + B_i + C_i + D_i.$$

Thus, the temperature at the current grid point is updated according to the Gauss–Seidel iteration

$$\widetilde{T}_{i,j} = \frac{A_i T_{i-1,j} + B_i T_{i+1,j} + C_i T_{i,j-1} + D_i T_{i,j+1}}{E_i}.$$

To improve the convergence rate of the iterative procedure, the Successive Over-Relaxation (SOR) method is applied. The corresponding iteration scheme can be expressed as

$$T_{i,j}^{(m+1)} = (1-\omega)T_{i,j}^{(m)} + \omega\widetilde{T}_{i,j},$$

where m is number of iteration, a ω is relaxation parameter. In the present study, the relaxation parameter was set to $\omega = 1.8$.

The iterations were repeated until the following convergence criterion was satisfied:

$$\max_{i,j} \left| T_{i,j}^{(m+1)} - T_{i,j}^{(m)} \right| < \varepsilon,$$

where ε denotes the prescribed convergence tolerance. In all numerical experiments presented in this study, the convergence tolerance was set to $\varepsilon = 10^{-6}$. This value ensured stable convergence of the iterative process while providing sufficient numerical accuracy for comparison with the analytical solution and the experimental measurements.

A convergence study was performed for the relaxation parameter ω in the range $1.0 \leq \omega \leq 2.0$ with an increment of 0.1. For each value of ω, the number of iterations required to satisfy the convergence criterion $\varepsilon = 10^{-6}$ was recorded. The results are summarized in Table 1. It can be observed that the number of iterations decreases as ω increases up to $\omega = 1.8$, while the iterative process becomes unstable and fails to converge for $\omega \geq 1.9$. Therefore, $\omega = 1.8$ was selected for all subsequent numerical computations.

Table 1 - Convergence study of the SOR method for different values of the relaxation parameter ω.

Parameter ω	**Iterations**	**Final residual, *K***	**Converged**	**Time, *sec***
1.0	3394	9.99×10^{-7}	Yes	18.45
1.1	2945	9.99×10^{-7}	Yes	15.94
1.2	2546	9.99×10^{-7}	Yes	13.76
1.3	2186	9.98×10^{-7}	Yes	12.10
1.4	1857	9.99×10^{-7}	Yes	10.26
1.5	1554	9.99×10^{-7}	Yes	8.58
1.6	1272	9.96×10^{-7}	Yes	6.98
1.7	1004	9.99×10^{-7}	Yes	5.48
1.8	**748**	$\mathbf{9.96 \times 10^{-7}}$	**Yes**	**4.15**
1.9	416	—	No	2.29
2.0	186	—	No	1.03

The Dirichlet boundary condition (2) is enforced directly at the inner boundary of the cylinder:

$$T_{0,j} = f.$$

At the interfaces $r = r_1$ and $r = r_2$ continuity conditions for temperature and heat flux are imposed. After applying the finite difference approximation, the temperature at the interface is determined from the following expressions

$$T_{N_1,j} = \frac{\frac{k_1}{\Delta r_1} T_{N_1-1,j} + \frac{k_2}{\Delta r_2} T_{N_1+1,j}}{\frac{k_1}{\Delta r_1} + \frac{k_2}{\Delta r_2}},$$

$$T_{N_1+N_2,j} = \frac{\frac{k_2}{\Delta r_2} T_{N_1+N_2-1,j} + \frac{k_3}{\Delta r_3} T_{N_1+N_2+1,j}}{\frac{k_2}{\Delta r_2} + \frac{k_3}{\Delta r_3}}.$$

At the outer boundary, heat exchange with the environment is described by a dimensionless Robin boundary condition. To simplify the formulation, the Biot number is defined as

$$Bi = \frac{hr_3}{k_3},$$

The finite difference approximation of the boundary condition (5) then takes the following form

$$T_{N,j} = \frac{T_{N-1,j} + Bi\Delta r_3 T_a(\varphi_j)}{1 + Bi\Delta r_3}.$$

The periodicity condition with respect to the angular coordinate is implemented as

$$T_{i,0} = T_{i,M}, \qquad T_{i,-1} = T_{i,M-1}.$$

A mesh sensitivity analysis was performed to evaluate the influence of spatial discretization on the numerical solution obtained by the successive over-relaxation (SOR) method. Three computational meshes, $(N_1, N_2, N_3, M) = (18, 18, 18, 64)$, $(20, 20, 20, 72)$, and $(25, 25, 25, 90)$, were was approximately ±0.3°C. Temperature data were recorded automatically at 10-minute intervals over an extended monitoring period. The experimental configuration, sensor arrangement, and measurement procedure are described in detail in [23].

The numerical model employed in the present study reproduces the same physical configuration and material arrangement. The measured temperatures at the inner and outer boundaries were imposed as boundary conditions, while the thermophysical properties of the three soil layers correspond to those used in the numerical simulations. The experimental measurements were used exclusively for validation of the computed temperature distribution by comparing the predicted and measured temperatures at the observation locations.

The validation data were obtained from an independent laboratory experiment reported in [23], ensuring that the comparison was performed using experimentally measured temperatures rather than synthetic or numerically generated data.

Experimental uncertainty

The experimental validation presented in this study is based on measurements obtained during a collaborative experimental campaign carried out within the same research project as the related investigations reported in [20, 21, 22, 23]. The experimental facility, measurement procedure, and data acquisition protocol were common to these studies, while the present work focuses on the analytical and numerical investigation of the steady-state heat conduction problem.

Temperature measurements were performed using thermocouples installed at predefined locations within the multilayer cylindrical system. The temperature measurements were performed using temperature sensors with a measurement uncertainty of $\pm 0.1^0 C$. The sensor position uncertainty was $\pm 0.5cm$, while the sensor response time was $0.75s$. Following the uncertainty analysis presented in the original experimental study, the total measurement uncertainty was evaluated as

$$\sigma_T = \sqrt{\sigma_m^2 + \sigma_r^2 + \sigma_t^2},$$

where σ_m denotes the sensor measurement uncertainty, σ_r is the uncertainty resulting from the sensor positioning, and σ_t represents the uncertainty associated with the finite sensor response time.

The position-related and temporal uncertainties were estimated as

$$\sigma_r = \frac{\partial T}{\partial r}\delta_r, \quad \sigma_t = \frac{\partial T}{\partial t}\delta_t,$$

where $\delta_r = 0.5cm$ is the uncertainty in the sensor location and $\delta_t = 0.75s$ is the sensor response time. The spatial temperature gradient was evaluated from the numerical solution, while the temporal derivative was estimated from the experimental measurements using finite-difference approximations.

Since the analytical and numerical models developed in the present work are validated against these experimental measurements, the observed discrepancies between the calculated and measured temperatures should be interpreted considering the inherent experimental uncertainty in addition to numerical discretization and model simplifications. The obtained agreement between the analytical solution, the SOR solution, and the experimental data remains within the expected uncertainty range of the experimental measurements, supporting the reliability of the proposed methodology.

Comparison of Temperature Fields

Figure 1 illustrates the temperature distribution obtained from the analytical solution, whereas Figure 2 presents the corresponding numerical results computed using the finite difference method and the SOR algorithm.

considered. For all simulations, the material properties, boundary conditions, relaxation parameter $\omega = 1.8$, and convergence criterion $\varepsilon = 10^{-6}$ were kept unchanged.

The numerical temperatures were compared at the representative radial locations $r = r_0$, $r = r_1$, $r = r_2$, and $r = r_3$ for $\varphi = 0$. The obtained results are summarized in Table 2. As the mesh was refined, only small changes in the numerical temperatures were observed. In particular, the maximum temperature difference between the selected mesh $(20, 20, 20, 72)$ and the finer mesh $(25, 25, 25, 90)$ was approximately 0.025 K, whereas the computational time increased from approximately 4.0 s to 8.6 s. Therefore, the mesh $(20, 20, 20, 72)$ was selected for all subsequent simulations because it provides a suitable balance between computational efficiency and numerical accuracy.

Table 2 - Mesh sensitivity analysis of the numerical solution obtained by the SOR method at $\varphi = 0$.

Mesh	Iterations	Time (s)	$T(r_0)$ (K)	$T(r_1)$ (K)	$T(r_2)$ (K)	$T(r_3)$ (K)
$18 \times 18 \times 18 \times 64$	658	2.860	298.052	291.2563	288.6471	287.4130
$20 \times 20 \times 20 \times 72$	748	3.990	298.052	291.2431	288.6401	287.4093
$25 \times 25 \times 25 \times 90$	1033	8.608	298.052	291.2183	288.6267	287.4024

The discretization of the governing boundary value problem results in a system of linear algebraic equations

$$\mathbf{AT} = \mathbf{F},$$

The resulting system is solved iteratively using the SOR method. The computed numerical solution is then used for comparison with the analytical solution and experimental measurements in order to evaluate the accuracy of the developed mathematical model.

Results and discussion

The accuracy of the proposed numerical method was assessed by comparing the SOR results with the analytical solution presented in Section **??**. To further validate the model, the analytical and numerical temperature distributions were also compared with experimental data obtained from the multilayer cylindrical system. The agreement between the calculated and measured temperatures provides an indication of the reliability of the proposed approaches.

Experimental methodology

The experimental data used for validation were obtained from the laboratory study described in [23]. The experimental setup consisted of a multilayer soil container with thermally insulated side walls to minimize lateral heat losses and ensure that heat transfer occurred predominantly in one direction. The total length of the container was 30 cm, with the interface between the two soil layers located at 15 cm. One end of the container was subjected to heating, while the opposite end exchanged heat with the surrounding air. Temperature measurements at both external boundaries were used to define the boundary conditions of the numerical model, whereas the temperature at the interface was used for validation and comparison with the numerical solution.

Five calibrated temperature sensors were installed along the container. Two sensors were positioned close to the external boundaries to record the ambient and boundary temperatures, while three sensors were embedded inside the soil, including one located exactly at the interface between the two layers. According to the sensor specifications, the measurement uncertainty

Both solutions reveal the same overall heat-transfer behavior. The highest temperatures are observed near the inner boundary, where the heating source is applied. Moving outward through the cylindrical layers, the temperature gradually decreases as heat is conducted across the materials and dissipated to the surrounding environment through convective heat exchange at the outer surface.

A visual comparison of the two temperature fields demonstrates a good level of agreement between the analytical and numerical results. The distributions remain smooth throughout the domain, with no visible oscillations or artificial discontinuities. This confirms the correct implementation of the interface conditions between the layers and indicates that the numerical scheme provides stable and reliable solutions.

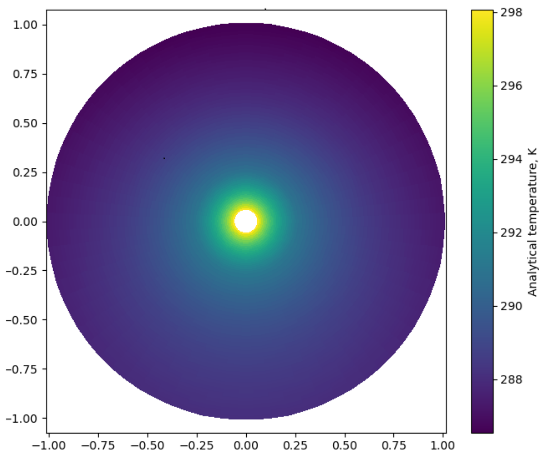

Figure 1 - Temperature distribution obtained from the analytical solution in the multilayer cylin-drical domain.

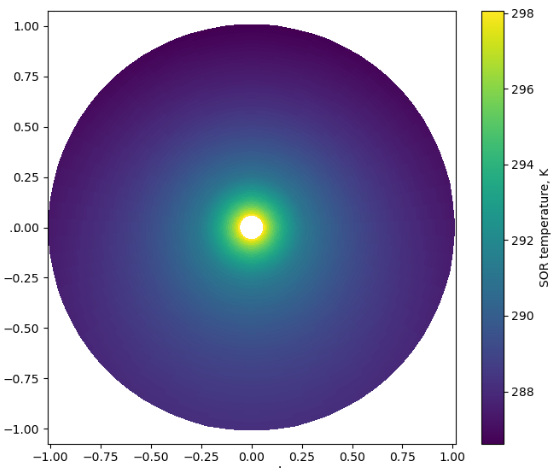

Figure 2 - Temperature distribution obtained using the finite difference method and the SOR algorithm.

Error Analysis Between the Analytical and Numerical Solutions

For a quantitative comparison of the two solutions, the absolute error was evaluated as

$$E(r, \varphi) = \left| T_{SOR}(r, \varphi) - T_A(r, \varphi) \right|, \tag{29}$$

where T_A and T_{SOR} denote the analytical and numerical solutions, respectively.

The spatial distribution of the absolute error is presented in Figure 3.

The highest differences between the analytical and numerical solutions occur in near the inner boundary, where the temperature changes most rapidly. In this region, even small discretization errors can lead to slightly larger deviations. However, as the distance from the heat source increases, the discrepancy quickly diminishes and remains very small throughout the rest of the domain. Overall, the results demonstrate that the numerical method is capable of accurately capturing the temperature distribution, including regions with pronounced temperature gradients.

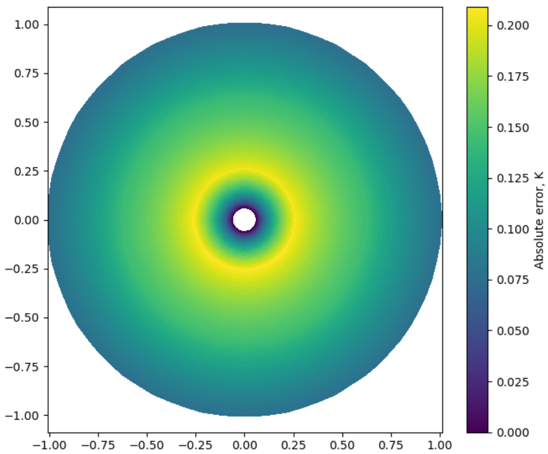

Figure 3 - Distribution of the absolute error between the analytical and numerical solutions.

For a quantitative evaluation of the numerical accuracy, the principal statistical characteristics of the error were calculated and are presented in Table 3.

Table 3 - Statistical error measures between the analytical and numerical solutions.

Metric	**Value**
Mean Absolute Error (K)	0.158568
Root Mean Square Error (K)	0.167308
Maximum Absolute Error (K)	0.241001
Mean Relative Error (%)	0.054654

Overall, the analytical and numerical solutions are in good agreement. The mean relative error remains below 0.05%, and the largest deviation observed in the domain is only 0.241 K. Such a good level of consistency demonstrates that the numerical model successfully captures the heat-transfer behavior of the system and confirms the suitability of the SOR-based finite difference approach for this class of problems.

Comparison of the Analytical Solution, SOR Results, and Experimental Data

To further assess the reliability of the proposed model, the radial temperature distributions obtained from the analytical solution and the numerical SOR method were compared with the experimental measurements. Since temperature data were available only along the direction $\varphi = 0$, the validation was performed for this angular position.

The comparison is shown in Figure 4 where the analytical and numerical temperature profiles are plotted together with the experimental data. The boundaries between the three layers are marked by vertical dashed lines at $r = r_1$ and $r = r_2$. This representation makes it possible to evaluate how accurately the proposed approaches reproduce the measured temperature distribution across the entire multilayer cylindrical domain.

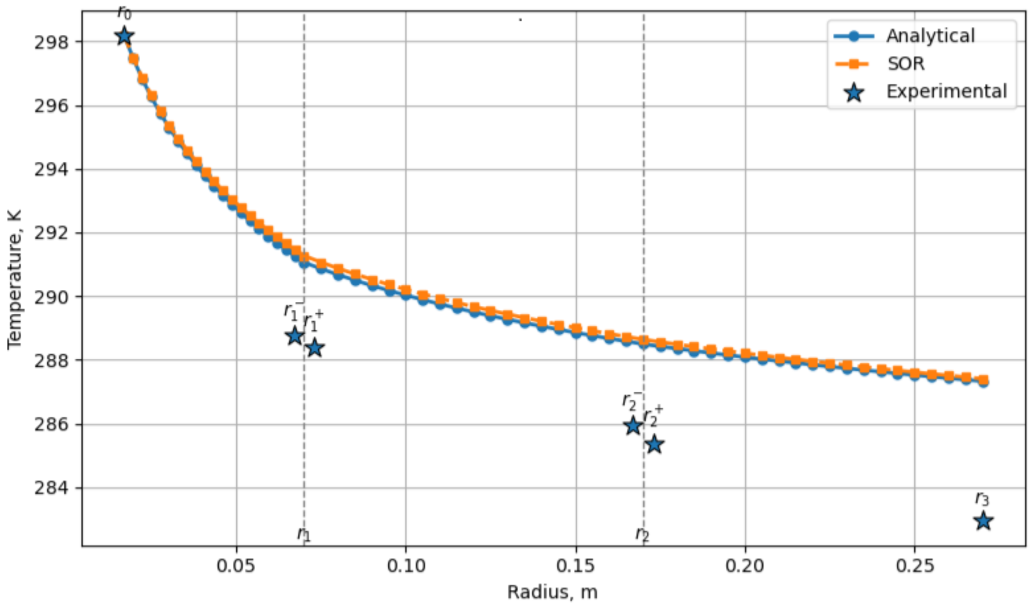

Figure 4 - Comparison of radial temperature profiles obtained from the analytical solution, the numerical SOR solution, and experimental measurements at $\varphi = 0$.

All three datasets follow the same general pattern: the temperature gradually decreases with increasing distance from the heated inner cylinder. The analytical and numerical results almost coincide over the entire domain, demonstrating that the finite difference model accurately reproduces the analytical solution.

Small changes in the slope of the temperature curves can be observed near the interfaces at $r = r_1$ and $r = r_2$. These changes are expected because the thermal conductivities of the adjacent layers differ. Slight deviations between the analytical and numerical solutions are most noticeable close to the interfaces and near the inner boundary, where the temperature gradients are the steepest.

The experimental measurements show good agreement with both calculated solutions and capture the same overall temperature trend. The largest differences between the measured and predicted temperatures occur near the layer boundaries. Such discrepancies may be related to factors that are not fully represented in the mathematical model, including thermal contact resistance, local material inhomogeneities, variations in thermophysical properties, and measurement errors.

To quantify the agreement between the model predictions and the experimental observations, the Mean Absolute Error (MAE) and the Root Mean Square Error (RMSE) were evaluated. The obtained values are summarized in Table 4.

The results presented in Table 4 show that the analytical solution is in slightly closer agreement with the experimental measurements than the numerical solution. The MAE for the analytical model is $2.872K$, compared to $3.024K$ for the SOR-based numerical approach. The same tendency can be observed for the RMSE values.

However, the difference between the two sets of errors is relatively small, remaining below 5%. This suggests that the error associated with the numerical discretization is minor in comparison with other sources of uncertainty, such as simplifications o f t he m athematical m odel, material variability, and measurement inaccuracies. Overall, the numerical solution reproduces the analytical results with acceptable accuracy and provides a realistic representation of the experimentally observed temperature field.

Table 4 - Comparison of the analytical and numerical models with respect to the experimental data.

Method	**MAE, K**	**RMSE, K**
Analytical Solution	2.872	3.105
SOR Solution	3.024	3.261

Discussion of results

This study investigated the steady-state heat conduction problem in a multilayer cylindrical domain using a finite difference method combined with the Successive Over-Relaxation (SOR) algorithm. The numerical results were compared with both the analytical solution and experimental temperature measurements.

A comparison between the analytical and numerical solutions showed a good agreement. The mean relative error remained below 0.054%, while the maximum deviation did not exceed $0.241K$. Such small differences indicate that the proposed discretization accurately captures the temperature field and that the SOR algorithm provides stable and efficient convergence.

A good agreement between the calculated and measured temperature distributions was observed. Although the analytical solution yielded slightly lower error values, the difference between the analytical and numerical results was small relative to the uncertainties associated with material properties, simplifying assumptions of the model, and measurement errors.

The close correspondence between the two solutions suggests that the numerical approximation introduces only a minor additional error. These results indicate that the finite difference–SOR approach is suitable for problems where analytical solutions are difficult to derive.

The developed numerical method can be used to model heat transfer in multilayer cylindrical structures. In particular, it offers a useful framework for the analysis of direct and inverse heat conduction problems in multilayer media with complex geometries and boundary conditions.

Conclusion

This paper considered a steady-state heat conduction problem in a multilayer cylindrical domain subjected to non-uniform boundary conditions. An analytical solution was obtained by applying the method of separation of variables together with a Fourier series representation. The developed solution describes the temperature field in each layer of the cylindrical structure while satisfying the continuity conditions at the interfaces between adjacent materials.

To solve the same problem numerically, a finite difference formulation of the Laplace equation in polar coordinates was constructed. The resulting system of algebraic equations was solved using the Successive Over-Relaxation (SOR) method. The numerical model accounts for the multilayer nature of the domain, variations in thermal conductivity between layers, and convective heat exchange at the outer surface. For the considered example, the iterative process converged after 748 iterations.

The numerical results were found to be in very close agreement with the analytical solution. The Mean Absolute Error (MAE) was $0.158K$, the Root Mean Square Error (RMSE) was $0.167K$, and the maximum deviation did not exceed 0.209 K. The mean relative error remained below

0.05%, confirming the good accuracy of the developed finite difference scheme and the correct implementation of the SOR method.

The developed models were also validated using experimental temperature measurements obtained from a multilayer cylindrical system. Both the analytical and numerical solutions captured the overall behavior of the measured temperature field and showed good agreement with the experimental data. The MAE relative to the measurements was $2.872K$ for the analytical solution and $3.024K$ for the numerical solution. The remaining differences can be explained by factors that are difficult to fully account for in the mathematical model, including thermal contact resistance at layer interfaces, material inhomogeneities, and experimental uncertainties.

The obtained results demonstrate that the SOR-based finite difference approach provides an accurate and computationally efficient tool for analyzing heat transfer in multilayer cylindrical structures. The method can be used in the study of thermal processes in pipelines, insulation systems, geothermal applications, and other engineering systems with layered cylindrical geometries.

Future research may be directed toward transient heat conduction problems, models with temperature-dependent material properties, and inverse problems aimed at identifying unknown thermophysical parameters from experimental observations.

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АНАЛИТИЧЕСКОЕ И ПРИБЛИЖЕННОЕ РЕШЕНИЕ ДВУМЕРНОГО УРАВНЕНИЯ ЛАПЛАСА В ЦИЛИНДРИЧЕСКИХ КООРДИНАТАХ

Аннотация

В данной работе исследуется стационарная задача теплопроводности в многослойной цилиндрической области с неоднородными граничными условиями. Рассмотрены два подхода к решению прямой задачи: аналитический метод, основанный на разделении переменных и разложении в ряд Фурье, и численный метод, основанный на конечно-разностной аппроксимации в сочетании с алгоритмом последовательной верхней релаксации (SOR). Получено точное аналитическое решение уравнения Лапласа в полярных координатах для трёхслойной цилиндрической системы с условиями сопряжения на границах раздела слоёв и конвективным теплообменом на внешней границе. Для численного решения разработана конечно-разностная схема, а возникающая система линейных алгебраических уравнений решается итерационным методом SOR. Точность численного подхода оценена путём сравнения с аналитическим решением и экспериментальными измерениями температуры. Полученные результаты демонстрируют отличное согласование аналитического и численного решений. Средняя абсолютная ошибка между двумя подходами составляет $0.158K$, среднеквадратическая ошибка — $0.167K$, а максимальная абсолютная ошибка не превышает $0.241K$. Экспериментальная валидация показала, что обе модели корректно воспроизводят общее распределение температуры в многослойной цилиндрической системе. Средняя абсолютная ошибка относительно экспериментальных данных составляет $2.872K$ для аналитического решения и $3.024K$ для решения, полученного методом SOR. Проведённый анализ подтверждает высокую точность и устойчивость разработанного конечно-разностного алгоритма и демонстрирует его применимость для моделирования процессов теплопереноса в многослойных цилиндрических конструкциях. Предложенный подход может быть использован при тепловом анализе трубопроводов, геотермальных систем, теплоизоляционных конструкций и других инженерных объектов, содержащих многослойные среды.

Ключевые слова: теплопроводность; уравнение Лапласа; многослойная цилиндрическая область; аналитическое решение; метод конечных разностей; метод последовательной верхней релаксации (SOR); распределение температуры; экспериментальная валидация; полярные координаты; тепловое моделирование.

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ЦИЛИНДРЛІК КООРДИНАТАЛАРДАҒЫ ЕКІ ӨЛШЕМДІ ЛАПЛАС ТЕҢДЕУІНІҢ АНАЛИТИКАЛЫҚ ЖӘНЕ САНДЫҚ ШЕШІМІ

Аңдатпа

Бұл жұмыста біртекті емес шекаралық шарттары бар көпқабатты цилиндрлік облыстағы стационарлық жылуөткізгіштік есебі зерттеледі. Тура есепті шешудің екі тәсілі қарастырылады: айнымалыларды жіктеу және Фурье қатарына жіктеу әдістеріне негізделген аналитикалық тәсіл және соңғы айырымдық аппроксимация мен тізбекті жоғарғы релаксация (SOR) алгоритмін пайдаланатын сандық тәсіл. Қабаттар арасындағы түйісу шарттары және сыртқы шекарадағы конвективтік жылуалмасу ескерілген үш қабатты цилиндрлік жүйе үшін полярлық координаталардағы Лаплас теңдеуінің дәл аналитикалық шешімі алынды. Сандық шешім үшін соңғы айырымдық схема құрылып, алынған сызықтық алгебралық теңдеулер жүйесі SOR итерациялық әдісімен шешілді. Сандық тәсілдің дәлдігі аналитикалық шешіммен және температураның эксперименттік өлшемдерімен салыстыру арқылы бағаланды. Алынған нәтижелер аналитикалық және сандық шешімдердің өте жақ-сы сәйкестігін көрсетті. Екі тәсіл арасындағы орташа абсолюттік қате $0.158K$, орташа квадраттық қате $0.167K$, ал максималды абсолюттік қате $0.241K$. Эксперименттік тексеру екі модельдің де көпқабатты цилиндрлік жүйедегі температураның жалпы таралуын дұрыс сипаттайтынын көрсетті. Эксперименттік деректерге қатысты орташа абсолюттік қате аналитикалық шешім үшін $2.872K$, ал SOR шешімі үшін $3.024K$ құрайды. Жүргізілген талдау ұсынылған соңғы айырымдық алгоритмнің жоғары дәлдігі мен орнықтылығын растайды және оның көпқабатты цилиндрлік құрылымдардағы жылуалмасу процестерін модельдеуде тиімді қолданылатынын көрсетеді. Ұсынылған тәсіл құбыржолдардың, геотермалдық жүйелердің, жылуоқшаулау құрылымдарының және көпқабатты орталарды қамтитын басқа да инженерлік объектілердің жылулық талдауында қолданылуы мүмкін.

Түйін сөздер: жылуөткізгіштік, Лаплас теңдеуі, көпқабатты цилиндрлік облыс, аналитикалық шешім, соңғы айырымдар әдісі, тізбекті жоғарғы релаксация әдісі (SOR), температураның таралуы, эксперименттік валидация, полярлық координаталар, жылулық модельдеу.