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ON A BOUNDARY VALUE PROBLEM FOR A LOADED FRACTIONAL DIFFERENTIAL EQUATION

Abstract

The paper investigates a two-point boundary value problem for a loaded differential equation with a Caputo fractional derivative. The Dzhumabaev parametrization method is applied to study the problem. Within this approach, the initial interval is divided into a finite number of subintervals, and additional parameters and new functions are introduced, which makes it possible to reduce the original boundary value problem to an equivalent multipoint boundary value problem with unknown parameters. The equivalence of the original and transformed problems is established. Based on the obtained integral representations, a system of algebraic equations with respect to the introduced parameters is constructed. Sufficient conditions for the existence and uniqueness of a solution are established and related to the invertibility of the corresponding matrix. An algorithm based on the method of successive approximations is developed to determine the unknown parameters and functions. A numerical example demonstrating the application of the proposed algorithm and confirming its effectiveness is presented. The obtained results demonstrate the efficiency of the parametrization method for investigating boundary value problems for loaded fractional-order differential equations and constructing their approximate solutions.

Keywords: problem for loaded differential equation, Caputo derivative, parametrization's method, approximate solution.

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Introduction

Models that incorporate fractional derivatives offer a more precise mathematical representation compared to classical integer-order models, as fractional differentiation and integration make it possible to account for memory and hereditary properties characteristic of many materials and processes [1]. This capability is a key advantage of fractional-order approaches, since such effects are typically beyond the scope of traditional differential equations of integer order. The advancement of fractional differential equation theory has significantly expanded their use in modeling a wide range of phenomena in science and engineering. In particular, these equations have been effectively applied in areas such as control theory, the physics of complex systems, neural modeling, electrochemistry, the mechanics of porous media, electromagnetism, and various other disciplines (see [2]-[6]). The growing interest in fractional calculus is reflected in numerous fundamental monographs and review papers [7]-[9], as well as in the references cited therein. Significant progress has been achieved in the study of initial and boundary value problems for fractional differential equations with Riemann–Liouville operators of order ($0 < \alpha \leq 1$) [10, 11]. In papers [12]-[15], certain classes of Cauchy problems for functional differential equations with fractional derivatives of both Riemann–Liouville and Caputo types of the same order were examined. Loaded differential equations occupy a distinctive position in the theory of fractional differential equations. One of the most general definitions of such equations was introduced by Nakhushev [16]. The rapid development of research in optimal control of agro-economic systems, as well as in problems related to groundwater level regulation and soil moisture management, has led to an increased interest in boundary value problems for this class of equations [17]-[19]. Important results in the theory of boundary value problems for loaded equations of parabolic, parabolic–hyperbolic, and elliptic–hyperbolic types were obtained in [20, 21]. This paper is devoted to the study of boundary value problems for loaded differential equations with a fractional derivative. The main objective is to establish conditions for the existence and uniqueness of solutions, as well as to develop numerical algorithms for their computation. Motivated by the above considerations, on the interval $[0,T]$ we consider the following linear boundary value problem for a loaded differential equation with a Caputo derivative of order $\alpha \in (0,1)$.

$$ {}_{C}D^{\alpha}x(t) = A(t)x(t) + \sum_{j=0}^{m} K_j(t)x(\xi_j) + f(t), \quad x \in \mathbb{R}^n, \quad 0 = \xi_0 < \xi_1 < \ldots, \xi_m = T, \qquad (1) $$

$$ Bx(0) + Cx(T) = d, \quad d \in \mathbb{R}^n, \qquad (2) $$

where $A(t)$, $K_j(t)$ $(j = 0,\ldots,m)$, and $f(t)$ are continuous on $[0,T]$.

Let $\mathbb{AC}([0,T],\mathbb{R}^n)$ is the space of absolutely continuous functions $x : [0,T] \to \mathbb{R}^n$ with the norm $\|x\|_1 = \max\limits_{t\in[0,T]} \|x(t)\|$.

A function $x^* : (0,T) \to \mathbb{R}^n$ is called a solution of (1), (2) if $x^* \in \mathbb{AC}([0,T],\mathbb{R}^n)$ satisfies Eq.(1) together with the condition (2).

Material and methods

The main purpose of this work is to apply the parameterization method for solving problem (1), (2). Based on this method, an algorithm for finding its solution will be constructed, and sufficient conditions for its solvability will be established.

Let $h_i = \xi_i - \xi_{i-1}$, $i = 1,\ldots,m$. We take a number $\ell \in \mathbb{N}$ and construct a partition the interval

$$ [0,T) = \bigcup_{r=1}^{m\ell} [t_{r-1}, t_r) $$

where $t_{i\ell} = \xi_i$, $i = 0, \ldots, m$, $\frac{h_1}{\ell} = t_r - t_{r-1}$, $r = 1, \ldots, \ell$, $\frac{h_2}{\ell} = t_r - t_{r-1}$, $r = \ell+1, \ldots, 2\ell$, and so on $\frac{h_m}{\ell} = t_r - t_{r-1}$, $r = (m-1)\ell + 1, \ldots, m\ell$.

Let us introduce the space $\mathbb{AC}\big([0,T], t_r, \mathbb{R}^{m\ell n}\big)$, consisting of vector-valued functions $x(t) = (x_1(t), \ldots, x_{m\ell}(t))$, where each function $x_r(t)$ is absolutely continuous on the interval $[t_{r-1}, t_r)$ and possesses a finite left-hand limit $\lim\limits_{t \to t_r^-} x_r(t)$, $r = 1, \ldots, m\ell$. This space is endowed with the norm $\|x[\cdot]\|_2 = \max\limits_{r=1,\ldots,m\ell} \sup\limits_{t \in [t_{r-1}, t_r)} \|x_r(t)\|$.

Then we denote the restriction of the function $x(t)$ by $x_r(t)$, i.e. $x_r(t) = x(t)$, $t \in [t_{r-1}, t_r)$, $r = 1, \ldots, m\ell$. In this case the problem (1), (2) is reduced to a multipoint boundary value problem

$$
{}_C D^{\alpha}_{t_{r-1}} x_r(t) = A(t)x_r(t) + \sum_{j=0}^{m-1} K_j(t) x_{j\ell+1}(\xi_j) + K_m(t) \lim_{t \to T^-} x_{m\ell}(t) + f^*(t),
$$
$$
t \in [t_{r-1}, t_r), \ r = 1, \ldots, m\ell, \tag{3}
$$

$$
Bx_1(0) + C \lim_{t \to T^-} x_{m\ell}(t) = d, \tag{4}
$$

$$
\lim_{t \to t_s^-} x_s(t) = x_{s+1}(t_s), \quad s = 1, \ldots, m\ell - 1. \tag{5}
$$

Here an equality (5) is conditions for matching the solution at the interior points and

$$
f^*(t) = f(t) - \frac{1}{\mathbf{\Gamma}(1-\alpha)} \sum_{i=1}^{r-1} \int\limits_{t_{r-1}}^{t_r} (t-\xi)^{-\alpha} x_i^{'}(\xi) d\xi.
$$

Let a solution to problem (3)-(5) is the system of functions $x^*[t] = (x_1^*(t), \ldots, x_{m\ell}^*(t))$, where $x^*[t] \in \mathbb{AC}([0,T], t_r, R^{m\ell n})$ with a Caputo derivative that satisfy the system (3) and conditions (4), (5).

Lemma 1. *The problems* (1),(2) *and* (3)-(5) *are equivalent.*

Дәлелдеме. Let $x^*(t)$ be a solution of problem (1),(2), and $x_r^*(t)$ be its restriction to the intervals $[t_{r-1}, t_r)$ for all $r = 1, \ldots, m\ell$. Then, from the absolute continuity of $x^*(t)$ on $[0,T]$, it follows that $x_r^*(t)$ is absolutely continuous on $[t_{r-1}, t_r)$ for all $r = 1, \ldots, m\ell$ and possesses finite left-hand limits

$$
\lim_{t \to t_r^-} x_r^*(t) = \lim_{t \to t_r^-} x^*(t) = x^*(t_r) = x_{r+1}^*(t_r), \quad r = 1, \ldots, m\ell.
$$

By virtue of (1), on each interval $[t_{r-1}, t_r)$ for all $r = 1, \ldots, m\ell$, the equality holds

$$
{}_C D^{\alpha}_{t_{r-1}} x_r^*(t) - A(t)x_r^*(t) - \sum_{j=0}^{m} K_j(t) x_{j\ell+1}^*(t_{j\ell}) - f^*(t) =
$$

$$
= {}_C D^{\alpha} x^*(t) - A(t)x^*(t) - \sum_{j=0}^{m} K_j(t) x^*(\xi_j) - f(t) = 0, \quad t \in [t_{r-1}, t_r), \ r = 1, \ldots, m\ell.
$$

That is, $x_r^*(t)$ is the restriction of the function $x^*(t)$ to the intervals $[t_{r-1}, t_r)$, $r = 1, \ldots, m\ell$, satisfies equation (3). Condit ion (4) follows from condition (2).

Conversely, let $\tilde{x}[t] \in \mathbb{AC}\big([0,T], t_r, \mathbb{R}^{m\ell n}\big)$ be a solution of (3)–(5), where $x\tilde{}[t] = (x\tilde{}_1(t), \ldots, x\tilde{}_{m\ell}(t))$. Define $\tilde{x} : [0,T] \to \mathbb{R}^n$ by $\tilde{x}(t) := \tilde{x}_r(t)$, $\quad t \in [t_{r-1}, t_r)$, $\quad r = 1, \ldots, m\ell$, and $\tilde{x}(T) := \lim\limits_{t \to T^-} \tilde{x}_{m\ell}(t)$.

From (4) and (5), it follows that $\tilde{x}(t)$ is absolutely continuous on $[0,T]$ and satisfies the boundary condition (2)

$$B\tilde{x}(0) + C\tilde{x}(T) = B\tilde{x}_1(0) + C\lim_{t\to T^-}\tilde{x}_{m\ell} = d.$$

Since the system of functions $\tilde{x}[t]$ is a solution of equation (3), the function $\tilde{x}(t)$ has a Caputo derivative and satisfies (1) for all t on $(0,T)$ except at the loading points; that is, there exists a function ${}_CD^{\alpha}_{t_{r-1}}x_r(t)$ for all $t\in[0,T]\backslash\{t_r\}$, $r=1,\dots,m\ell$.

Since

$$\lim_{t\to t_r^-}{}_CD^{\alpha}_{t_{r-1}}\tilde{x}_r(t) = A(t_r)\lim_{t\to t_r^-}\tilde{x}_r(t) + \sum_{j=0}^{m}K_j(t_r)\tilde{x}_{j\ell+1}(t_{j\ell}) + f^*(t_r) =$$

$$= A(t_r)\tilde{x}_{r+1}(t) + \sum_{j=0}^{m}K_j(t_r)\tilde{x}_{j\ell+1}(t_{j\ell}) + f^*(t_r) = {}_CD^{\alpha}\tilde{x}_{r+1}(t_r),\ r=1,\dots,m\ell,$$

it follows from (5) that ${}_CD^{\alpha}x(t)$ exists for all $t\in[0,T]$ and satisfies equation (1). □

By introducing the parameters $\lambda_r := x_r(t_{r-1})$, $r=1,\dots,m\ell$, $\lambda_{m\ell+1} := \lim_{t\to T^-}x_{m\ell}(t)$, and substituting $u_r(t) = x_r(t)-\lambda_r$, $t\in[t_{r-1},t_r)$, $r=1,\dots,m\ell$, we transform the boundary value problem (3)-(5) into the equivalent multi-point boundary value problem with parameters

$${}_CD^{\alpha}_{t_{r-1}}u_r(t) = A(t)(u_r(t)+\lambda_r) + \sum_{j=0}^{m}K_j(t)\lambda_{j\ell+1} + f^*(t), \quad t\in[t_{r-1},t_r), \tag{6}$$

$$u_r(t_{r-1}) = 0, \quad r=1,\dots,m\ell, \tag{7}$$

$$B\lambda_1 + C\lambda_{m\ell+1} = d, \tag{8}$$

$$\lambda_s + \lim_{t\to t_s^-}u_s(t) = \lambda_{s+1}(t_s), \quad s=1,\dots,m\ell. \tag{9}$$

A solution to the problem (6)-(9) is a pair $(\lambda^*, u^*[t])$ with components $\lambda^* = (\lambda_1^*,...,\lambda_{m\ell+1}^*)\in\mathbb{R}^{n(m\ell+1)}$, $u^*[t] = (u_1^*(t),...,u_{m\ell}^*(t))\in\mathbb{AC}([0,T],t_r,R^{m\ell n})$, that satisfy (6)-(9).

If $x^*(t)$ is a solution to problem (1), (2), then the pair $(\lambda^*, u^*[t])$ with components $\lambda^* = (\lambda_1^*,...,\lambda_{m\ell+1}^*)\in\mathbb{R}^{n(m\ell+1)}$, $u^*[t] = (u_1^*(t),...,u_{m\ell}^*(t))\in\mathbb{AC}([0,T],t_r,R^{m\ell n})$, where $\lambda_r^* = x^*(t_{r-1})$, $u_r^*(t) = x^*(t)-x^*(t_{r-1})$, $t\in[t_{r-1},t_r)$, $r=1,\dots,m\ell$, satisfies to problem (1), (2). Conversely, if the pair $(\tilde{\lambda},\tilde{u}[t])$ is a solution to problem (6)-(9), then the function $\tilde{x}(t)$ defined by the equalities $\tilde{x}(t) = \tilde{\lambda}_r + \tilde{u}_r(t)$, $t\in[t_{r-1},t_r)$, $r=1,\dots,m\ell$, and $\tilde{x}(T) = \tilde{\lambda}_{m\ell+1}$ is a solution to origin problem (1), (2).

On interval $[t_{r-1},t_r)$, $r=1,\dots,m\ell$, the initial value problem (6), (7) to the equivalent system of integral equations

$$u_r(t) = \frac{1}{\mathbf{\Gamma}(\alpha)}\int_{t_{r-1}}^{t}(t-\tau)^{\alpha-1}A(\tau)u_r(\tau)d\tau + \tag{10}$$

$$+\frac{1}{\mathbf{\Gamma}(\alpha)}\int_{t_{r-1}}^{t}(t-\tau)^{\alpha-1}\Big[A(\tau)\lambda_r + \sum_{j=0}^{m}K_j(\tau)\lambda_{j\ell+1} + f^*(\tau)\Big]d\tau.$$

By taking the limit as $t\to t_r^-$ in the right-hand side of (10) and substituting the resulting expressions into condition (9), as well as multiplying equation (8) by h_m^{α}, we obtain a system of equations with respect to the unknown parameters

$$Bh_m^{\alpha}\lambda_1 + Ch_m^{\alpha}\lambda_{m\ell+1} = h_m^{\alpha}d, \tag{11}$$

$$\big(I_n + \mathcal{D}_s(A,t_s)\big)\lambda_s - \lambda_{s+1} + \sum_{j=0}^{m} \mathcal{H}_s(K_j,t_s)\lambda_{j\ell+1} = -\mathrm{f}_s(t_s) - \mathrm{g}_s(u,t_s), \tag{12}$$

where for all $s = 1,\dots,m\ell$

$$\mathcal{D}_s(A,t_s) = \frac{1}{\mathbf{\Gamma}(\alpha)}\int\limits_{t_{s-1}}^{t_s}(t_s-\tau)^{\alpha-1}A(\tau)d\tau, \quad \mathcal{H}_s(K_j,t_s) = \frac{1}{\mathbf{\Gamma}(\alpha)}\int\limits_{t_{s-1}}^{t_s}(t_s-\tau)^{\alpha-1}K_j(\tau)d\tau, \quad j=\overline{0,m},$$

$$\mathrm{f}_s(t_s) = \frac{1}{\mathbf{\Gamma}(\alpha)}\int\limits_{t_{s-1}}^{t_s}(t_s-\tau)^{\alpha-1}f(\tau)d\tau, \qquad \mathrm{g}_1(u,t_s) = \frac{1}{\mathbf{\Gamma}(\alpha)}\int\limits_{t_0}^{t_1}(t_1-\tau)^{\alpha-1}A(\tau)u_1(\tau)d\tau,$$

$$\mathrm{g}_s(u,t_s) = \frac{1}{\mathbf{\Gamma}(\alpha)}\int\limits_{t_{s-1}}^{t_s}(t_s-\tau)^{\alpha-1}A(\tau)\Big(u_s(\tau) + \frac{1}{\mathbf{\Gamma}(1-\alpha)}\sum_{i=1}^{s-1}\int\limits_{t_{i-1}}^{t_i}(\tau-\xi)^{-\alpha}u_i^{'}(\xi)d\xi\Big)d\tau, \ \ s=\overline{2,m\ell}.$$

The matrix corresponding to the left part of the system (11), (12) will be denoted by $Q_\alpha(\ell)$. Let us write this system in the following form

$$Q_\alpha(\ell)\lambda = -\mathcal{F}(\ell) - \mathcal{G}(u,\ell), \quad \lambda \in \mathbb{R}^{n(m\ell+1)}, \tag{13}$$

with the right sides of the view

$$\mathcal{F}(\ell) = (-h_m^\alpha d, \mathrm{f}_1(t_1),\dots,\mathrm{f}_{m\ell}(T)), \quad \mathcal{G}(u,\ell) = (0,\mathrm{g}_1(u,t_1),\dots,\mathrm{g}_{m\ell}(u,T)).$$

Results and discussion

If the matrix $Q_\alpha(\ell)$ is invertible, then to find the pair $(\lambda, u[t])$ we use the method of successive approximations, which is carried out according to the following algorithm.

Step 0. (a) Provided that, for the chosen $\ell \in \mathbb{N}$, the matrix $Q_\alpha(\ell)$ is invertible, the initial approximation with respect to the parameter $\lambda^{(0)} = (\lambda_1^{(0)},\dots,\lambda_{m\ell+1}^{(0)})$ is obtained from the equation $Q_\alpha(\ell)\lambda = -F(\ell)$, that is,

$$\lambda^{(0)} = -[Q_\alpha(\ell)]^{-1}\mathcal{F}(\ell);$$

(b) Using the components of the vector $\lambda^{(0)} \in \mathbb{R}^{n(m\ell+1)}$ we solve the initial value problem (6), (7) with $\lambda_r = \lambda_r^{(0)}$, $r = 1,\dots,m\ell$, on each interval $[t_{r-1},t_r)$. As a result, we obtain the functions $u_r^{(0)}(t)$, $r=1,\dots,m\ell$.

Step 1. (a) Substituting the obtained functions $u_r^{(0)}(t)$, $r=1,\dots,m\ell$, into the right-hand side of (13), we determine the next approximation $\lambda^{(1)}$ from the equation

$$Q_\alpha(\ell)\lambda = -F(\ell) - \mathcal{G}(u^{(0)},\ell);$$

(b) On each interval $[t_{r-1},t_r)$, the initial value problem (6), (7) are then solved with $\lambda_r = \lambda_r^{(1)}$, yielding the functions $u_r^{(1)}(t)$, $r=1,\dots,m\ell$.

And so on. Continuing the process, at the k-th step we obtain a system of pairs $(\lambda^{(k)}, u_r^{(k)}[t])$, $k = 0,1,2,\dots$.

We introduce the following auxiliary definitions and theorem

Definition 2. *(Mittag–Leffler Function, [22]). Let $n > 0$. The function E_n defined by*

$$E_n(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(jn+1)}, \qquad z \in \mathbb{C},$$

whevener the series converges is called the Mittag–Leffler function of order n.

This function has been introduced by Mittag-Leffler. We immediately notice that

$$E_1(z) := \sum_{j=0}^{\infty} \frac{z^j}{\mathbf{\Gamma}(j+1)} = \sum_{j=0}^{\infty} \frac{z^j}{j!} = e^z$$

is just the well known exponential function.

Definition 3. *Let $n_1, n_2 > 0$. The function E_{n_1,n_2} defined by*

$$E_{n_1,n_2}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\boldsymbol{\Gamma}(jn_1+n_2)},$$

whenever the series converges is called the two-parameter Mittag-Leffler function with parameters n_1 and n_2.

Theorem 4. *(Theorem 2.2, [23]) Let $a(t)$ and $g(t)$ be nondecreasing functions, $g(t) \leq C$ for all $t \in [0,T]$, and let $0 < \beta < 1$. If $x \in L_+^{\infty}[0,T]$ satisfies the inequality*

$$x(t) \leq a(t) + g(t) \int_0^t (t-s)^{\beta-1} x(s)\, ds, \quad t \in [0,T],$$

then

$$x(t) \leq a(t) E_\beta \left[g(t) \boldsymbol{\Gamma}(\beta)(t^\beta) \right],$$

where E_β denotes the Mittag–Leffler function.

Let us introduce the notations

$$\sigma = \max_{t\in[0,T]} \|A(t)\|, \quad \kappa = \max_{j=\overline{0,m}} \max_{t\in[0,T]} \|K_j(t)\|, \quad h = \max\{h_1, ..., h_m\}.$$

Theorem 5. *Let the matrix $Q_\alpha(\ell) : \mathbb{R}^{n(m\ell+1)} \to \mathbb{R}^{n(m\ell+1)}$ be invertible and suppose that the following inequalities hold:*

1. $\|[Q_\alpha(\ell)]^{-1}\| \leq \gamma(\ell)$,
2. $q(\ell) = \dfrac{\gamma(\ell)}{\boldsymbol{\Gamma}^{\,2}(1+\alpha)} E_\alpha\Big(\sigma \boldsymbol{\Gamma}(\alpha)\Big(\dfrac{h}{\ell}\Big)^\alpha\Big)(\sigma + \kappa(m+1))^2 \Big(\dfrac{h}{\ell}\Big)^{2\alpha} < 1.$

Then the sequence of pairs $(\lambda^{(k)}, u^{(k)}[t])$ converges to $(\lambda^*, u^*[t])$ as $k \to \infty$.

Дәлелдеме. Under the assumptions of the theorem, we define $\lambda^{(0)}$ at the zeroth step of the algorithm. Then, the following estimate holds:

$$\|\lambda^{(0)}\| = \max_{r=\overline{1,m\ell+1}} \|\lambda_r^{(0)}\| \leq \gamma(\ell)\|\mathcal{F}(\ell)\| \leq \frac{\gamma(\ell)}{\mathbf{\Gamma}(\alpha+1)} \max(\|d\|, \|f\|_1)\Big(\frac{h}{\ell}\Big)^\alpha \tag{14}$$

Let $u_r^{(0)}(t)$ be defined as (10) at $\lambda_r = \lambda_r^{(0)}$. Applying Grönwall's inequality, we obtain

$$\|u_r^{(0)}(t)\| \leq \sup_{t\in[t_{r-1},t_r)} E_\alpha\Big(\sigma(t-t_{r-1})\Big)\frac{1}{\mathbf{\Gamma}(\alpha)}\left\|\int\limits_{t_{r-1}}^{t}(t-\tau)^{\alpha-1}\Big[A(\tau)\lambda_r^{(0)}+\sum_{j=0}^{m}K_j(\tau)\lambda_{j\ell+1}+f^*(\tau)\Big]d\tau\right\| \leq$$

$$\leq \frac{1}{\mathbf{\Gamma}(\alpha+1)}E_\alpha\Big(\sigma\Big(\frac{h}{\ell}\Big)^\alpha\Big)\Big(\sigma\|\lambda_r\|+\kappa(m+1)\|\lambda_{j\ell+1}\|+\sup_{t\in[t_{r-1},t_r)}\|f^*\|\Big)\Big(\frac{h}{\ell}\Big)^\alpha$$

Taking into account (14), we obtain

$$\|u^{(0)}[\cdot]\|_2 = \max_{r=\overline{1,m\ell}}\sup_{t\in[t_{r-1},t_r)}\|u_r^{(0)}(t)\| \leq$$

$$\leq \frac{1}{\mathbf{\Gamma}(\alpha+1)}E_\alpha\Big(\sigma\mathbf{\Gamma}(\alpha)\Big(\frac{h}{\ell}\Big)^\alpha\Big)\left(\frac{\gamma(\ell)}{\mathbf{\Gamma}(\alpha+1)}\big(\sigma+\kappa(m+1)\big)\Big(\frac{h}{\ell}\Big)^{2\alpha}+1\right)\max(\|d\|,\|f\|_1) = P(\ell).$$

After determining $\lambda^{(1)}$ from an equation $Q_\alpha(\ell)\lambda = -F(\ell)-\mathcal{G}(u^{(0)},\ell)$, we estimate the difference $\lambda^{(1)}-\lambda^{(0)}$:

$$\|\lambda^{(1)}-\lambda^{(0)}\| = \|[Q_\alpha(\ell)]^{-1}\|\|\mathcal{G}(u^{(0)},\ell)\| \leq \gamma(\ell)\max_{s=1,\dots,m\ell}\left\|\frac{1}{\mathbf{\Gamma}(\alpha)}\int\limits_{t_{s-1}}^{t_s}(t_s-\tau)^{\alpha-1}A(\tau)u_r^{(0)}(\tau)d\tau\right\| \leq$$

$$\leq \frac{\gamma(\ell)}{\mathbf{\Gamma}(1+\alpha)}\big(\sigma+\kappa(m+1)\big)\Big(\frac{h}{\ell}\Big)^\alpha\|u^{(0)}[\cdot]\|_2 \leq \frac{\gamma(\ell)P(\ell)}{\mathbf{\Gamma}(1+\alpha)}\big(\sigma+\kappa(m+1)\big)\Big(\frac{h}{\ell}\Big)^\alpha.$$

Having obtained $u_r^{(1)}(t)$ as the solution of the Cauchy problem (10) for $\lambda_r = \lambda_r^{(1)}$, let us estimate the following difference

$$\|u_r^{(1)}(t)-u_r^{(0)}(t)\| \leq \frac{\sigma}{\mathbf{\Gamma}(\alpha)}\int\limits_{t_{r-1}}^{t}(t-\tau)^{\alpha-1}\|u_r^{(1)}(\tau)-u_r^{(0)}(\tau)\|d\tau+$$

$$+\frac{\sigma+\kappa(m+1)}{\mathbf{\Gamma}(\alpha)}\int\limits_{t_{r-1}}^{t}(t-\tau)^{\alpha-1}d\tau\|\lambda_r^{(1)}-\lambda_r^{(0)}\|.$$

Therefore, by the fractional Grönwall inequality, we have

$$\|u_r^{(1)}(t)-u_r^{(0)}(t)\| \leq \frac{\sigma+\kappa(m+1)}{\mathbf{\Gamma}(1+\alpha)}E_\alpha\Big(\sigma\mathbf{\Gamma}(\alpha)\Big(\frac{h}{\ell}\Big)^\alpha\Big)\Big(\frac{h}{\ell}\Big)^\alpha\|\lambda^{(1)}-\lambda^{(0)}\|,$$

that is,

$$\|u_r^{(1)}[\cdot]-u_r^{(0)}[\cdot]\|_2 \leq \frac{\gamma(\ell)P(\ell)}{\mathbf{\Gamma}^2(1+\alpha)}E_\alpha\Big(\sigma\mathbf{\Gamma}(\alpha)\Big(\frac{h}{\ell}\Big)^\alpha\Big)(\sigma+\kappa(m+1))^2\Big(\frac{h}{\ell}\Big)^{2\alpha}.$$

Proceeding with the iterative process, we obtain a sequence of pairs $(\lambda^{(k)},u^{(k)}[t])$. The following estimates hold for the differences $u^{(k)}(t)-u^{(k-1)}(t)$ and $\lambda^{(k)}-\lambda^{(k-1)}$:

$$\|u_r^{(k)}[\cdot]-u_r^{(k-1)}[\cdot]\|_2 \leq \frac{\sigma+\kappa(m+1)}{\mathbf{\Gamma}(1+\alpha)}E_\alpha\Big(\sigma\mathbf{\Gamma}(\alpha)\Big(\frac{h}{\ell}\Big)^\alpha\Big)\Big(\frac{h}{\ell}\Big)^\alpha\|\lambda^{(k)}-\lambda^{(k-1)}\|, \qquad (15)$$

$$\|\lambda^{(k+1)}-\lambda^{(k)}\| \leq \frac{\gamma(\ell)P(\ell)}{\mathbf{\Gamma}^2(1+\alpha)}E_\alpha\Big(\sigma\mathbf{\Gamma}(\alpha)\Big(\frac{h}{\ell}\Big)^\alpha\Big)(\sigma+\kappa(m+1))^2\Big(\frac{h}{\ell}\Big)^{2\alpha}\|\lambda^{(k)}-\lambda^{(k-1)}\|. \qquad (16)$$

Since $q(\ell) < 1$ by the assumptions of the theorem 2, it follows from inequalities (15) and (16) that the sequence $\lambda^{(k)}$ converges to λ^*, and the sequence of functions system $u_r^{(k)}[t]$ converges to the function system $u^*[t]$ in the norm of the space $\mathbb{AC}\big([0,T], t_r, \mathbb{R}^{m\ell n}\big)$. $\square$

Numerical experiment

We consider on $[0,1]$ the boundary value problem for a loaded fractional differential equation

$$ {}_C D^{\alpha} x(t) = t^2 x(t) - x(0.25) + x(0.75) - 0.5t^2, \quad x \in \mathbb{R}, \tag{17}$$

$$ x(0) + x(1) = 1, \tag{18}$$

where $\alpha(t) = 0.4$, $t \in [0, 0.5)$ and $\alpha(t) = 0.8$, $t \in [0.5, 1)$.

The exact solution to the problem at hand is the function $x^*(t) = 0.5$.

We partition the given interval as follows $[0,1) = \bigcup\limits_{r=1}^{4} [0.5(r-1), 0.5r)$ and introduce parameters

$$ \lambda_1 = x(0), \quad \lambda_2 = x(0.25), \quad \lambda_3 = x(0.5), \quad \lambda_4 = x(0.75), \quad \lambda_5 = \lim_{t \to 1^-} x(t).$$

Consider the following substitution $u_r(t) = x(t) - \lambda_r$ for all $r = 1,2,3,4$. Then the test problem (17), (18) reduces to a problem with parameters

$$ {}_C D^{\alpha_r} u_r(t) = t^2 (u_r(t) + \lambda_r) - \lambda_2 + \lambda_4 + f^*(t), \quad t \in [0.5(r-1), 0.5r), \tag{19}$$

$$ u_r(0.5(r-1)) = 0, \quad r = 1,2,3,4, \tag{20}$$

$$ \lambda_1 + \lambda_5 = 1, \tag{21}$$

$$ \lambda_s + \lim_{t \to 0.25s^-} u_s(t) = \lambda_{s+1}, \quad s = 1,2,3,4, \tag{22}$$

where

$$ \alpha_1, \alpha_2 = 0.4, \quad \alpha_3, \alpha_4 = 0.8, \quad f^*(t) = -0.5t^2, \quad t \in [0, 0.25),$$

$$ f^*(t) = -0.5t^2 - \frac{1}{\mathbf{\Gamma}(1-\alpha)} \sum_{i=1}^{r-1} \int\limits_{0.25(r-1)}^{0.25r} (\tau - \xi)^{-\alpha_r} u_i'(\xi) d\xi, \quad t \in [0.25(r-1), 0.25r), \ r = 2,3,4.$$

For fixed parameters, the solution to the initial problem (19), (20) is given by

$$ u_1(t) = \frac{1}{\mathbf{\Gamma}(0.4)} \int\limits_0^t (t-\tau)^{-0.6} \tau^2 u_1(\tau) d\tau + \frac{2\lambda_1 - 1}{\mathbf{\Gamma}(17/5)} t^{12/5} - \frac{\lambda_2 - \lambda_4}{\mathbf{\Gamma}(7/5)} t^{2/5}, \ t \in [0, 0.25), \tag{23}$$

$$ u_2(t) = \frac{1}{\mathbf{\Gamma}(0.4)} \int\limits_{0.25}^t (t-\tau)^{-0.6} \tau^2 u_2(\tau) d\tau + \frac{2\lambda_2 - 1}{\mathbf{\Gamma}(17/5)} (t - 0.25)^{12/5} - \frac{\lambda_2 - \lambda_4}{\mathbf{\Gamma}(7/5)} (t-0.25)^{2/5} - \tag{24}$$

$$ - \frac{1}{\mathbf{\Gamma}(0.4)} \int\limits_{0.25}^t (t-\tau)^{-0.6} {}_C D^{0.4} u_1'(0.25, \tau) d\tau, \quad t \in [0.25, 0.5),$$

$$ u_3(t) = \frac{1}{\mathbf{\Gamma}(0.8)} \int\limits_{0.5}^t (t-\tau)^{-0.2} \tau^2 u_3(\tau) d\tau + \frac{2\lambda_3 - 1}{\mathbf{\Gamma}(19/5)} (t-0.5)^{14/5} - \frac{\lambda_2 - \lambda_4}{\mathbf{\Gamma}(9/5)} (t-0.5)^{4/5} - \tag{25}$$

$$-\sum_{s=1}^{2}\frac{1}{\mathbf{\Gamma}(0.4)}\int_{0.5}^{t}(t-\tau)^{-0.2}{}_C D^{0.8}_{\frac{s-1}{4}}u_s'(0.25s,\tau)d\tau,\quad t\in[0.5,0.75),$$

$$u_4(t)=\frac{1}{\mathbf{\Gamma}(0.8)}\int_{0.75}^{t}(t-\tau)^{-0.2}\tau^2u_4(\tau)d\tau+\frac{2\lambda_4-1}{\mathbf{\Gamma}(19/5)}(t-0.75)^{14/5}-\frac{\lambda_2-\lambda_4}{\mathbf{\Gamma}(9/5)}(t-0.75)^{4/5}- \tag{26}$$

$$-\sum_{s=1}^{3}\frac{1}{\mathbf{\Gamma}(0.4)}\int_{0.75}^{t}(t-\tau)^{-0.6}{}_C D^{0.8}_{\frac{s-1}{4}}u_s'(0.25s,\tau)d\tau,\quad t\in[0.5,0.75),$$

From (23)-(26) we determine $\lim\limits_{t\to 0.25r^-}u_r(t)$, $r=1,2,3,4$. Based on (21), (22), we construct the system of linear equations for unknown parameters

$$Q_\alpha\lambda=-\mathcal{F}_\alpha-\mathcal{G}_\alpha(u),\quad \lambda\in\mathbb{R}^5. \tag{27}$$

Here

$$Q_\alpha=\begin{pmatrix}1 & 0 & 0 & 0 & 1\\ 1+2b & -1-a & 0 & a & 0\\ 0 & 1+2b+c & -1 & a & 0\\ 0 & -c & 1+2d & -1+c & 0\\ 0 & -c & 0 & 1+2d+c & -1\end{pmatrix}$$

$$\mathcal{F}_\alpha=(1,b,b,d,d),\quad \mathcal{G}_\alpha(u)=\big(0,g_1(u,0.25),g_2(u,0.5),g_3(u,0.75),g_4(u,1)\big),$$

where

$$a=\frac{1}{\mathbf{\Gamma}\left(7/5\right)}\left(0.25\right)^{2/5},\ b=\frac{1}{\mathbf{\Gamma}\left(17/5\right)}\left(0.25\right)^{12/5},\quad c=\frac{1}{\mathbf{\Gamma}\left(9/5\right)}\left(0.25\right)^{4/5},\ d=\frac{1}{\mathbf{\Gamma}\left(19/5\right)}\left(0.25\right)^{14/5},$$

$$\mathrm{g}_1(u,0.25)=\frac{1}{\mathbf{\Gamma}\left(0.4\right)}\int_0^{0.25}\left(0.25-\tau\right)^{-3/5}\tau^2u_1(\tau)\,d\tau,$$

$$\mathrm{g}_2(u,0.5)=\frac{1}{\mathbf{\Gamma}\left(0.4\right)}\int_{0.25}^{0.5}\left(0.5-\tau\right)^{-3/5}\left[\tau^2u_2(\tau)-{}_C D^{0.4}u_1'(0.25,\tau)\right]d\tau,$$

$$\mathrm{g}_3(u,0.75)=\frac{1}{\mathbf{\Gamma}\left(0.8\right)}\int_{0.5}^{0.75}\left(0.75-\tau\right)^{-1/5}\left[\tau^2u_3(\tau)-\sum_{s=1}^{2}{}_C D^{0.8}_{\frac{s-1}{4}}u_s'(0.25s,\tau)\right]d\tau,$$

$$\mathrm{g}_4(u,1)=\frac{1}{\mathbf{\Gamma}\left(0.8\right)}\int_{0.75}^{1}\left(1-\tau\right)^{-1/5}\left[\tau^2u_4(\tau)-\sum_{s=1}^{3}{}_C D^{0.8}_{\frac{s-1}{4}}u_s'(0.25s,\tau)\right]d\tau.$$

The results of implementing the proposed algorithm are as follows:

Step 0. $\lambda_r^{(0)}=0.5$, for all $r=1,\ldots,5$, and $u_r^{(0)}(t)=0$, $t\in[0.25(r-1),0.25r)$, $r=1,\ldots,4$.

Step 1. $\lambda_r^{(1)}=0.5$, for all $r=1,\ldots,5$, and $u_r^{(1)}(t)=0$, $t\in[0.25(r-1),0.25r)$, $r=1,\ldots,4$.

And so on. Then $x(t)=0.5$, $t\in[0,1]$. All computations were performed in Spyder (Python 3.13).

Conclusion

In this work, an analysis of a two-point boundary value problem for a loaded differential equation with a fractional derivative is carried out. Within the framework of this study, the development of the parametrization method is presented, which makes it possible to reduce the original problem to an equivalent multipoint boundary value problem by introducing additional parameters. Based on the obtained formulation, the construction of approximate solutions is performed using an iterative procedure. The proposed algorithm provides successive refinement of the solution and allows one to establish sufficient conditions for the existence of a solution as well as for the convergence of the constructed iterative process. The obtained results demonstrate the effectiveness of the developed parametrization method for studying boundary value problems of the considered class. To ensure the continuity of the present research in this direction, it is planned in future work to prove that the obtained sufficient conditions for the solvability of the problem (1), (2) are also necessary.

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О КРАЕВОЙ ЗАДАЧЕ ДЛЯ НАГРУЖЕННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С ДРОБНОЙ ПРОИЗВОДНОЙ

Аннотация

В работе исследуется двухточечная краевая задача для нагруженного дифференциального уравнения с дробной производной Капуто. Для исследования поставленной задачи применяется метод параметризации Джумабаева. В рамках данного подхода исходный промежуток разбивается на конечное число частей, вводятся дополнительные параметры и новые функции, что позволяет свести исходную краевую задачу к эквивалентной многоточечной краевой задаче с неизвестными параметрами. Устанавливается эквивалентность исходной и преобразованной задач. На основе полученных интегральных представлений строится система алгебраических уравнений относительно введённых параметров. Устанавливаются достаточные условия существования и единственности решения, связанные с обратимостью соответствующей матрицы. Для нахождения неизвестных параметров и функций разработан алгоритм, основанный на методе последовательных приближений. Приведён численный пример, демонстрирующий применение предложенного алгоритма и подтверждающий его эффективность. Полученные результаты показывают эффективность метода параметризации при исследовании краевых задач для нагруженных дифференциальных уравнений дробного порядка и построении их приближённых решений.

Ключевые слова: задача для нагруженного дифференциального уравнения, производная Капуто, метод параметризации, приближенное решение.

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ЖҮКТЕМЕЛІ БӨЛШЕКТІК ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУ ҮШІН ШЕТТІК ЕСЕП ТУРАЛЫ

Аңдатпа

Жұмыста Капутоның бөлшек туындысы бар жүктелген дифференциалдық теңдеу үшін екі нүктелі шеттік есеп зерттеледі. Қойылған есепті зерттеу үшін Джумабаевтың параметрлеу әдісі қолданылады. Осы тәсіл аясында бастапқы аралық ақырлы санды бөліктерге бөлініп, қосымша параметрлер мен жаңа функциялар енгізіледі. Бұл бастапқы шеттік есепті белгісіз параметрлері бар оған эквивалентті көп нүктелі шеттік есепке келтіруге мүмкіндік береді. Бастапқы және түрлендірілген есептердің эквиваленттілігі белгіленеді. Алынған интегралдық өрнектер негізінде енгізілген параметрлерге қатысты алгебралық теңдеулер жүйесі құрылады. Сәйкес матрицаның қайтымдылығына байланысты шешімнің бар болуы мен жалғыздығының жеткілікті шарттары белгіленеді. Белгісіз параметрлер мен функцияларды анықтау үшін тізбекті жуықтау әдісіне негізделген алгоритм әзірленеді. Ұсынылған алгоритмнің қолданылуын көрсететін және оның тиімділігін растайтын сандық мысал келтіріледі. Алынған нәтижелер бөлшек ретті жүктелген дифференциалдық теңдеулерге арналған шеттік есептерді зерттеуде және олардың жуық шешімдерін құруда параметрлеу әдісінің тиімділігін көрсетеді.

Түйін сөздер: жүктелген дифференциалдық теңдеу үшін есеп, Капуто туындысы, параметрлеу әдісі, жуық шешім.