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ON THE CONTROL PROBLEM FOR A ONE-DIMENSIONAL PSEUDO-PARABOLIC EQUATION WITH A NONLOCAL BOUNDARY CONDITION

Abstract

This paper studies a control problem for a one-dimensional pseudo-parabolic equation with a nonlocal boundary condition of the Bitsadze-Samarskii type. The control function appears as a distributed source term on the right-hand side of the equation, and the objective is to steer the system from the zero initial state so that the weighted average temperature follows a prescribed trajectory. The equation contains a third-order mixed derivative term that accounts for memory or viscoelastic effects in heat conduction, which is particularly relevant for materials with internal structure such as polymers, biological tissues, and granular media. Using spectral decomposition and a biorthogonal basis, the problem is reduced to a Volterra integral equation of the first kind. The kernel of this integral equation is proved to be positive and continuous. Under suitable assumptions on the weight function and the spatial distribution of the control, the existence of an admissible control is established for a class of desired trajectories with sufficiently small norm. The Laplace transform method is employed to solve the integral equation and to obtain bounds on the control function. The results quantify the influence of the nonlocal boundary condition and the memory parameter on the controllability of the system.

Keywords: pseudo-parabolic equation, boundary value problem, Volterra integral equation, admissible control, thin rod.

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Introduction

Pseudo-parabolic equations arise in various applied contexts, including fluid flow in porous media, heat transfer in heterogeneous materials, and radiative diffusion. These models extend classical parabolic equations by incorporating higher-order mixed derivatives that capture memory, relaxation, and inertial effects. Unlike the standard heat equation, pseudo-parabolic models can describe finite-speed propagation of thermal disturbances, a phenomenon often referred to as the second sound effect, which is particularly relevant in materials with internal structure such as polymers, biological tissues, and granular media.

In recent years, growing interest in control theory and applied mathematics has spurred extensive research on control problems for pseudo-parabolic equations. Foundational contributions include the study of control and its comparison with classical parabolic models [1], and investigations on stability, uniqueness, and existence of solutions for various pseudo-parabolic systems [2]. Point control for parabolic and pseudo-parabolic models is discussed in detail in [3].

Control problems for parabolic equations have attracted considerable attention over the past decades due to their central role in modeling heat conduction and diffusion processes. One of the earliest contributions in this direction was made by Friedman [4], who studied optimal control problems for parabolic equations using variational methods. Fundamental results on the controllabi-lity of linear second-order parabolic equations were obtained by Fattorini and Russell [5], where the notions of exact and approximate controllability were first introduced.

The extension of control theory to infinite-dimensional systems was carried out by Egorov [6], who generalized Pontryagin's maximum principle to evolution equations in Banach spaces and established a bang-bang principle under natural assumptions. Comprehensive treatments of optimal control for distributed parameter systems, including parabolic equations, can be found in the classical monographs by Lions [8] and Fursikov [7].

The time-optimal control problem for the heat equation and the structure of optimal controls were intensively studied in subsequent works. In particular, the time-varying bang-bang property of optimal controls and its applications were analyzed in [9]. The bang-bang principle for time-optimal boundary control problems was further examined in [10], where it was shown that bang-bang controls remain optimal for arbitrary target temperature distributions.

The boundary control problem for the heat transfer equation in two-dimensional domain was considered in [15], where the admissibility of the control function was established using geometric estimates. Boundary control problems for one- and two-dimensional heat equations were further investigated in [16, 17], while the time-optimal control of a fourth-order parabolic equation in a two-dimensional domain was analyzed in [18].

Initial studies of time-optimal control problems for heat conduction equations in bounded n-dimensional domains were presented in [11], where sharp estimates for the minimal time required to achieve a prescribed average temperature were derived. Mathematical models of thermocontrol processes were analyzed in [12]. Various related problems for nonlinear and degenerate parabolic equations have also attracted considerable attention; see, for instance, [13], where local existence and uniqueness of weak solutions were established via the Galerkin method and the Banach fixed point theorem, and [14], where singular parabolic equations with logarithmic nonlinearities were studied.

Recently, in [19], a time-optimal control problem for the heat equation involving an involution operator in a multidimensional parallelepiped domain was investigated. In that work, an optimal estimate for the minimal time required to heat the domain to a prescribed average temperature was obtained. Recent developments have also addressed controllability issues from a spectral and observability viewpoint. In [20], spectral inequalities for the Laplace-Beltrami operator were derived, leading to observability and null-controllability results for the spherical heat equation. Boundary control problems for generalized heat equations involving Schr¨odinger-type operators were studied in [21]. Numerical aspects of boundary control and discretization schemes for heat equation were analyzed in [22]. Applications to fluid dynamics and thermally coupled systems were considered in [23].

In [24], time-optimal control problems for semilinear parabolic equations subject to Dirichlet boundary control and terminal state constraints are analyzed. In that work, a Hamiltonian-type functional is introduced in order to derive necessary optimality conditions with respect to the terminal time T. The existence of time-optimal controls for a class of semilinear parabolic equations with distributed control acting on a subdomain is investigated in [25], where sufficient conditions ensuring the attainability of the target state in minimal time are established.

For controllability of pseudo-parabolic equations, Zhou [26] analyzed approximate controllability and minimal time control problems for a pseudo-parabolic equation with memory term. Control problems for pseudo-parabolic equations have been studied in works [27, 28], where the control

In this paper, we study a control problem for a one-dimensional pseudo-parabolic equation with a nonlocal boundary condition of the Bitsadze–Samarskii type. The control function appears as a distributed source term on the right-hand side of the equation, and the goal is to steer the system from the zero initial state so that the weighted average temperature follows a prescribed trajectory. We first introduce the concept of a generalized solution for the initial-boundary value problem. Using spectral decomposition, we expand the solution in terms of the eigenfunctions of the corresponding nonlocal spectral problem. By introducing a suitable class of weight functions, we derive a Volterra integral equation of the first kind for the unknown control function. The necessary estimates for the kernel are established. Using these estimates, we prove the existence of an admissible control for a suitable class of desired trajectories. The results quantify the influence of the nonlocal boundary condition and the memory parameter on the controllability of the system.

To the best of our knowledge, control problems for pseudo-parabolic equations with Bitsadze-Samarskii type nonlocal boundary conditions have received little attention in the existing literature. Unlike the available works devoted to pseudo-parabolic equations under classical or different boundary conditions, the present paper considers the nonlocal condition $u(1,t) = u\big(\frac{1}{2},t\big)$, which leads to a non-orthogonal system of eigenfunctions and substantially complicates the corresponding spectral analysis. To overcome this difficulty, we employ a biorthogonal basis constructed from the adjoint spectral problem. This approach enables us to derive a Volterra integral equation of the first kind with a positive kernel and to establish the existence of an admissible control. Therefore, the novelty of the present work lies in the combination of the pseudo-parabolic model, the Bitsadze–Samarskii type nonlocal boundary condition, and the biorthogonal spectral approach used to solve the control problem.

Materials and methods

In this paper we consider the pseudo-parabolic equation

$$u_t(x,t) - u_{xx}(x,t) - \alpha\, u_{xxt}(x,t) = h(x)\,\mu(t), \quad x \in (0,1), \quad t > 0, \tag{1}$$

with a nonlocal boundary condition

$$u(0,t) = 0, \qquad u(1,t) = u\big(\tfrac{1}{2},t\big)\,, \quad t \geq 0, \tag{2}$$

and the homogeneous initial condition

$$u(x,0) = 0, \quad 0 \leq x \leq 1, \tag{3}$$

where $h(x)$ is a given function describing the spatial distribution of the control action, $\mu(t)$ is the control function, and $\alpha > 0$ is a constant characterizing the relaxation time of the thermal flux.

Equation (1) is a third-order pseudo-parabolic equation. The term αu_{xxt} accounts for memory or viscoelastic effects in the heat conduction process. Such models arise in the theory of viscoelasticity, fluid flow in fissured rocks, and heat conduction in heterogeneous media. When $\alpha = 0$, equation (1) reduces to the classical heat equation. For $\alpha > 0$, the equation describes finite-speed propagation of thermal disturbances, a phenomenon known as the second sound effect, which cannot be captured by the classical parabolic model.

From a physical point of view, equation (1) describes the evolution of the temperature field $u(x,t)$ in a thin homogeneous rod of unit length. The initial condition (3) means that the rod is initially at zero temperature. The right-hand side $h(x)\mu(t)$ represents a distributed heat source, where $\mu(t)$ is the time-dependent intensity and $h(x)$ is its spatial profile. The function $\mu(t)$ is the control input.

The boundary conditions are as follows. At the left end we have a fixed zero temperature, $u(0,t) = 0$. The right end is coupled with the interior point $x = \frac{1}{2}$ through the nonlocal condition

$u(1,t) = u(\frac{1}{2},t)$. Physically, such a condition may arise in feedback control systems where a sensor located at the midpoint $x = \frac{1}{2}$ regulates the temperature at the right end $x = 1$, or in applications requiring temperature equalization at two distinct points. The specific choice of the coupling point as $x = \frac{1}{2}$ introduces a natural geometric symmetry, simplifying the spectral analysis while preserving the nonlocal nature of the problem. This nonlocal constraint introduces spectral anomalies and controllability challenges that are absent in classical local boundary value problems.

Definition 1 *A control function $\mu(t)$ is called* admissible *if $\mu(t) \in L^2(\mathbb{R}_+)$ and $|\mu(t)| \le 1$ for all $t \ge 0$.*

We consider the spectral problem associated with the spatial operator of equation (1):

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < 1, \tag{4}$$

with the nonlocal boundary conditions

$$X(0) = 0, \quad X(1) = X(1/2). \tag{5}$$

For $\lambda \le 0$, problem (4)–(5) admits only the trivial solution. Hence, we consider $\lambda > 0$. In this case, the spectrum consists of two countable sequences of eigenvalues:

$$\lambda_{k,1} = \left(\frac{2(2k-1)\pi}{3}\right)^2, \qquad \lambda_{k,2} = (4k\pi)^2, \quad k \in \mathbb{N}. \tag{6}$$

The corresponding eigenfunctions are

$$X_{k,1}(x) = \sin\big(\sqrt{\lambda_{k,1}}\, x\big), \qquad X_{k,2}(x) = \sin\big(\sqrt{\lambda_{k,2}}\, x\big), \quad k \in \mathbb{N}.$$

The system $\{X_{k,j}\}$ is not orthogonal. Following [29], we introduce the adjoint problem:

$$Y''(x) + \lambda Y(x) = 0, \quad x \in (0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1),$$

with boundary conditions

$$Y(0) = 0, \quad Y(1) = 0,$$

and transmission conditions at $x = 1/2$:

$$Y(1/2+0) = Y(1/2-0), \qquad Y'(1/2+0) - Y'(1/2-0) = Y'(1).$$

The eigenvalues of the adjoint problem are the same as in (6). The adjoint eigenfunctions are given explicitly by

$$Y_{k,1}(x) = \begin{cases} \dfrac{8}{3}\sin\big(\sqrt{\lambda_{k,1}}x\big), & 0 \le x < \frac{1}{2}, \\ -\dfrac{4}{3\cos\sqrt{\lambda_{k,1}}}\sin\big(\sqrt{\lambda_{k,1}}(1-x)\big), & \frac{1}{2} < x \le 1, \end{cases}$$

$$Y_{k,2}(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2}, \\ -\dfrac{4}{\cos\sqrt{\lambda_{k,2}}}\sin\big(\sqrt{\lambda_{k,2}}(1-x)\big), & \frac{1}{2} < x \le 1. \end{cases}$$

The systems $\{X_{k,j}\}$ and $\{Y_{k,j}\}$ satisfy the biorthogonality condition

$$\int_0^1 X_{k,j}(x) Y_{\ell,m}(x)\, dx = \delta_{k\ell}\delta_{jm},$$

and are complete and biorthogonal in $L_2(0,1)$ (see [29]).

We define the class of weight functions as follows:

$$U := \left\{ \rho \in W_2^1(0,1) \;\middle|\; \begin{array}{l} \rho(x) \geq 0, \quad \int_0^1 \rho(x)\,dx = 1, \\ \rho_{k,j} \geq 0, \quad k \in \mathbb{N},\ j = 1,2, \end{array} \right\},$$

where $\rho_{k,j}$ are the Fourier coefficients of $\rho(x)$ with respect to the adjoint eigenfunctions $Y_{k,j}$, i.e.,

$$\rho(x) = \sum_{j=1}^{2}\sum_{k=1}^{\infty} \rho_{k,j} Y_{k,j}(x), \qquad \rho_{k,j} = \int_0^1 \rho(x) X_{k,j}(x)\,dx.$$

The class U is non-empty. Indeed, consider the function

$$\rho(x) = 2(1-x).$$

Clearly $\rho(x) \geq 0$ on $[0,1]$, $\int_0^1 \rho(x)\,dx = 1$, and $\rho \in W_2^1(0,1)$. Integrating by parts, for every k, j with $\omega = \sqrt{\lambda_{k,j}}$,

$$\rho_{k,j} = \int_0^1 2(1-x)\sin(\omega x)\,dx = \frac{2(\omega - \sin\omega)}{\omega^2}.$$

Since $\sin t \leq t$ for all $t \geq 0$, it follows that $\omega - \sin\omega \geq 0$ for every $\omega > 0$, and hence $\rho_{k,j} \geq 0$ for all k, j. Thus $\rho \equiv 2(1-x) \in U$, and the class U is non-empty.

The main objective of this work is to investigate the following control problem.

Control Problem. *For a given function $f(t)$, find an admissible control $\mu(t)$ such that the solution $u(x,t)$ of the initial-boundary value problem* (1)–(3) *satisfies the integral condition*

$$\int_0^1 \rho(x) u(x,t)\,dx = f(t), \quad t \geq 0, \tag{7}$$

where $\rho \in U$ is a given weight function.

This formulation represents a physically meaningful requirement that the weighted temperature in the rod follows a prescribed trajectory $f(t)$ over time. Physically, the integral condition (7) can be interpreted as the total heat content (or thermal energy) stored in the rod, weighted by the function $\rho(x)$. For example, if $\rho(x)$ represents the specific heat capacity distribution or a sensor sensitivity profile, then the integral gives a measurable output of the system. The control problem consists of finding $\mu(t)$ such that this measured output exactly matches the desired function $f(t)$. Unlike pointwise temperature constraints, this averaged condition characterizes a global thermal performance criterion. Such a formulation is particularly relevant in applications where only the overall thermal state of the system matters, for example in thermal processing of materials, temperature control in manufacturing, or heating systems. From a control-theoretic perspective, the integral constraint allows greater flexibility in the temperature distribution while still guaranteeing the required total thermal effect. The goal is to determine the control function $\mu(t)$ that realizes the desired weighted temperature dynamics $f(t)$.

For any constant $M > 0$, we denote by $W(M)$ the set of functions $f \in W_2^2(-\infty, +\infty)$ such that $f(t) = 0$ for all $t \leq 0$ and

$$\|f\|_{W_2^2(\mathbb{R}_+)} \leq M.$$

The main result of this section is the following theorem.

Theorem 2 *There exists $M > 0$ such that for every $f \in W(M)$ the solution $\mu(t)$ of the equation* (7) *exists and satisfies $|\mu(t)| \leq 1$.*

In the subsequent analysis, we will also require the following assumption on the spatial distribution $h(x)$ of the control action.

The Fourier coefficients of $h(x)$ with respect to the adjoint eigenfunctions $Y_{k,j}$ are nonnegative, i.e.,

$$h_{k,j} = \int_0^1 h(x)Y_{k,j}(x)\,dx \geq 0, \qquad j = 1,2,\ k \in \mathbb{N}.$$

This assumption is essential for ensuring the positivity of certain kernel functions and the solvability of the Volterra integral equation that will be derived in the next sections.

Results and discussion

In this section we reduce the control problem to a Volterra integral equation of the first kind. The reduction uses spectral decomposition and the biorthogonal basis introduced above.

Let $T > 0$ be arbitrary. We denote by $C([0,T];B)$ the space of continuous functions from $[0,T]$ into a Banach space B.

We introduce the subspace of $W_2^1(0,1)$ that incorporates the nonlocal boundary condition:

$$V := \left\{v \in W_2^1(0,1)\ :\ v(0) = 0,\ v(1) = v(1/2)\right\}.$$

Definition 3 *A function $u(x,t)$ is called a* generalized solution *of problem* (1)–(3) *if $u \in C([0,T];V)$ satisfies the weak formulation*

$$\begin{aligned}\frac{d}{dt}\big(u(\cdot,t),\varphi\big)_{L_2(0,1)} + \alpha\,\frac{d}{dt}\big(u_x(\cdot,t),\varphi_x\big)_{L_2(0,1)} + \big(u_x(\cdot,t),\varphi_x\big)_{L_2(0,1)}\\ = \mu(t)\big(h,\varphi\big)_{L_2(0,1)}\end{aligned} \tag{8}$$

for every test function $\varphi \in H_0^1(0,1)$ and almost every $t \in (0,T)$, together with the initial condition $u(x,0) = 0$.

The choice $\varphi \in H_0^1(0,1)$ ensures that the boundary terms arising from integration by parts vanish, since $\varphi(0) = \varphi(1) = 0$. The nonlocal condition $u(1,t) = u(1/2,t)$ is imposed on the solution u, which belongs to the space V.

Let $\{X_{k,j}\}$ and $\{Y_{k,j}\}$ be the biorthogonal systems defined in Section 2. We expand the solution $u(x,t)$ in the eigenfunctions $\{X_{k,j}\}$:

$$u(x,t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty} u_{k,j}(t)X_{k,j}(x), \qquad u_{k,j}(t) = \int_0^1 u(x,t)Y_{k,j}(x)\,dx.$$

The spatial distribution $h(x)$ of the control is expanded using the adjoint eigenfunctions $\{Y_{k,j}\}$:

$$h_{k,j} = \int_0^1 h(x)Y_{k,j}(x)\,dx, \qquad j = 1,2,\ k \in \mathbb{N}.$$

Substituting these expansions into (8) and using biorthogonality, we obtain for each k,j:

$$(1+\alpha\lambda_{k,j})u_{k,j}'(t) + \lambda_{k,j}u_{k,j}(t) = \mu(t)h_{k,j}, \quad u_{k,j}(0) = 0.$$

Solving this linear ODE gives

$$u_{k,j}(t) = \frac{h_{k,j}}{1+\alpha\lambda_{k,j}}\int_0^t e^{-\frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}}(t-s)}\mu(s)\,ds.$$

Thus,

$$u(x,t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{h_{k,j}}{1+\alpha\lambda_{k,j}}\left(\int_0^t e^{-\frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}}(t-s)}\mu(s)\,ds\right)X_{k,j}(x). \tag{9}$$

Lemma 4 *Let $\mu \in L^2(0,T)$ and $h \in L^2(0,1)$. Then the function $u(x,t)$ defined by* (9) *is a generalized solution of problem* (1)–(3) *in the sense of Definition 3.*

Proof of Lemma 4. We first show that $u \in C([0,T];V)$. Differentiating (9) with respect to x gives

$$u_x(x,t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{h_{k,j}}{1+\alpha\lambda_{k,j}} \left(\int_0^t e^{-\frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}}(t-s)} \mu(s)\, ds \right) X'_{k,j}(x).$$

Using the identity

$$\|X'_{k,j}\|^2_{L_2} = \lambda_{k,j}\|X_{k,j}\|^2_{L_2} = \lambda_{k,j}A_{k,j}, \qquad A_{k,j} = \int_0^1 X^2_{k,j}(x)\, dx,$$

which follows by integration by parts from the eigenvalue equation $X''_{k,j} = -\lambda_{k,j}X_{k,j}$, together with the regularizing effect of the factor $(1+\alpha\lambda_{k,j})^{-1}$, we obtain

$$\|u_x(\cdot,t)\|^2_{L_2} \le C \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{h^2_{k,j}}{(1+\alpha\lambda_{k,j})^2} \|X'_{k,j}\|^2_{L_2} \left(\int_0^t e^{-\frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}}(t-s)} \mu(s)\, ds \right)^2.$$

Since $A_{k,j} = \int_0^1 X^2_{k,j}(x)\, dx$ satisfies $0 < A_{k,j} \le \frac{1}{2}$ for all k, j, it follows that

$$\|u_x(\cdot,t)\|^2_{L_2} \le C \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{h^2_{k,j}\lambda_{k,j}A_{k,j}}{(1+\alpha\lambda_{k,j})^2} \left(\int_0^t e^{-\frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}}(t-s)} \mu(s)\, ds \right)^2.$$

Applying the Cauchy-Schwarz inequality to the time integral,

$$\begin{aligned} \left(\int_0^t e^{-\frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}}(t-s)} \mu(s)\, ds \right)^2 &\le \int_0^t e^{-2\frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}}(t-s)} ds \cdot \int_0^t |\mu(s)|^2 ds \\ &\le \frac{1+\alpha\lambda_{k,j}}{2\lambda_{k,j}} \|\mu\|^2_{L_2(0,T)}. \end{aligned}$$

Substituting this estimate,

$$\|u_x(\cdot,t)\|^2_{L_2} \le \frac{C}{2}\|\mu\|^2_{L_2(0,T)} \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{h^2_{k,j}A_{k,j}}{1+\alpha\lambda_{k,j}}.$$

Since $h \in L_2(0,1)$, its Fourier coefficients satisfy

$$\sum_{j=1}^{2}\sum_{k=1}^{\infty} h^2_{k,j}A_{k,j} = \|h\|^2_{L_2} < \infty.$$

Moreover, $\dfrac{1}{1+\alpha\lambda_{k,j}} \le 1$, so the series converges. Hence $\|u_x(\cdot,t)\|_{L_2}$ is uniformly bounded on $[0,T]$. The same argument applied to u (using $|X_{k,j}(x)| \le 1$) shows that $\|u(\cdot,t)\|_{L_2}$ is also bounded. Continuity in t follows from the dominated convergence theorem, because each term is continuous and the series converges uniformly. Thus $u \in C([0,T];H^1(0,1))$.

To justify the time derivatives in the weak formulation, differentiating (9) with respect to time t gives us

$$u_t(x,t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{h_{k,j}}{1+\alpha\lambda_{k,j}} \left[-\sigma_{k,j} \int_0^t e^{-\sigma_{k,j}(t-s)} \mu(s)\, ds + \mu(t) \right] X_{k,j}(x),$$

where $\sigma_{k,j} = \frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}} \le 1/\alpha$. Similarly, differentiating u_x with respect to t gives

$$u_{xt}(x,t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{h_{k,j}}{1+\alpha\lambda_{k,j}} \left[-\sigma_{k,j}\int_0^t e^{-\sigma_{k,j}(t-s)}\mu(s)\,ds + \mu(t)\right] X'_{k,j}(x).$$

Since $\sigma_{k,j} \le 1/\alpha$ and $\|X'_{k,j}\|^2_{L_2} = \lambda_{k,j}A_{k,j}$, the same convergence argument as for u_x shows that both series converge in $L^2(0,T;H^{-1}(0,1))$. Hence, the weak formulation (8) is justified.

Each eigenfunction $X_{k,j}$ satisfies $X_{k,j}(0) = 0$ and $X_{k,j}(1) = X_{k,j}(1/2)$. Therefore $u(0,t) = 0$ and $u(1,t) = u(1/2,t)$, which means $u \in C([0,T];V)$.

Finally, substituting the series (9) into the weak formulation (8) with test functions $\varphi \in H_0^1(0,1)$ and using biorthogonality verifies that u satisfies the equation. The initial condition $u(x,0) = 0$ follows from (9) because the integrals vanish at $t = 0$. This completes the proof. □

By substituting the expression (9) into the integral condition with weight function $\rho \in U$, we obtain

$$\begin{aligned} f(t) &= \int_0^1 \rho(x)\,u(x,t)\,dx \\ &= \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{h_{k,j}}{1+\alpha\lambda_{k,j}} \left(\int_0^t e^{-\frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}}(t-s)}\mu(s)\,ds\right) \int_0^1 \rho(x)\,X_{k,j}(x)\,dx. \end{aligned}$$

The Fourier coefficients of the weight function ρ with respect to $X_{k,j}$ are defined as

$$\rho_{k,j} := \int_0^1 \rho(x)X_{k,j}(x)\,dx.$$

Thus

$$f(t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{h_{k,j}\,\rho_{k,j}}{1+\alpha\lambda_{k,j}} \int_0^t e^{-\frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}}(t-s)}\mu(s)\,ds.$$

Define the kernel

$$K(t) := \sum_{j=1}^{2}\sum_{k=1}^{\infty} \Lambda_{k,j}\,e^{-\frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}}t}, \quad t \ge 0, \tag{10}$$

where

$$\Lambda_{k,j} = \frac{h_{k,j}\,\rho_{k,j}}{1+\alpha\lambda_{k,j}}, \quad j = 1,2, \quad k \in \mathbb{N}.$$

Then we arrive at the Volterra integral equation of the first kind

$$\int_0^t K(t-\tau)\,\mu(\tau)\,d\tau = f(t), \quad t > 0. \tag{11}$$

The solvability of (11) depends on the properties of $K(t)$. We now establish the necessary estimates.

Lemma 5 *For any $\rho \in U$, the Fourier coefficients*

$$\rho_{k,j} = \int_0^1 \rho(x)X_{k,j}(x)\,dx$$

satisfy

$$0 \le \rho_{k,j} \le \frac{C}{k}, \qquad j = 1,2,\ k \in \mathbb{N},$$

with $C > 0$ independent of k and j.

Proof of Lemma 5. The inequality $\rho_{k,j} \geq 0$ follows from the definition of U (recall that $\rho_{k,j} \geq 0$ is part of the definition). For the upper bound, we start from

$$\rho_{k,j} = \int_0^1 \rho(x) \sin(\sqrt{\lambda_{k,j}} x)\, dx.$$

Integrating by parts:

$$\int_0^1 \rho(x) \sin(\sqrt{\lambda_{k,j}} x)\, dx = -\frac{1}{\sqrt{\lambda_{k,j}}} \int_0^1 \rho'(x) \cos(\sqrt{\lambda_{k,j}} x)\, dx + \frac{\rho(1) \cos \sqrt{\lambda_{k,j}} - \rho(0)}{\sqrt{\lambda_{k,j}}}.$$

Taking absolute values and using $|\cos(\sqrt{\lambda_{k,j}} x)| \leq 1$:

$$\left| \int_0^1 \rho(x) \sin(\sqrt{\lambda_{k,j}} x)\, dx \right| \leq \frac{1}{\sqrt{\lambda_{k,j}}} \int_0^1 |\rho'(x)|\, dx + \frac{|\rho(1)| + |\rho(0)|}{\sqrt{\lambda_{k,j}}}.$$

Since $\rho \in W_2^1(0,1)$, the integral $\int_0^1 |\rho'(x)| dx$ is finite, and $\rho(0), \rho(1)$ are bounded. Hence there exists $C_1 > 0$ such that

$$\int_0^1 |\rho'(x)| dx + |\rho(0)| + |\rho(1)| \leq C_1.$$

Thus

$$\left| \int_0^1 \rho(x) \sin(\sqrt{\lambda_{k,j}} x)\, dx \right| \leq \frac{C_1}{\sqrt{\lambda_{k,j}}}.$$

Now $\sqrt{\lambda_{k,j}} \sim \pi k$, so $1/\sqrt{\lambda_{k,j}} \sim 1/(\pi k)$. Therefore,

$$\rho_{k,j} \leq \frac{C}{k}.$$

□

Lemma 6 *The kernel function $K(t)$ defined by (10) is positive and continuous on $[0, \infty)$.*

Proof of Lemma 6. From the definition,

$$K(t) = \sum_{j=1}^{2} \sum_{k=1}^{\infty} \Lambda_{k,j} e^{-\frac{\lambda_{k,j}}{1+\alpha \lambda_{k,j}} t},$$

where $\Lambda_{k,j} = \dfrac{h_{k,j} \rho_{k,j}}{1 + \alpha \lambda_{k,j}}$. By assumptions, $h_{k,j} \geq 0$, $\rho_{k,j} \geq 0$, and $\lambda_{k,j} > 0$. Hence $\Lambda_{k,j} \geq 0$.

Since h and ρ are not identically zero and their Fourier coefficients are nonnegative, we may assume without loss of generality that $h_{1,1} > 0$ and $\rho_{1,1} > 0$. (If $h_{1,1} = 0$, we take the next smallest index where the coefficients are positive.) Therefore $\Lambda_{1,1} > 0$, and

$$K(t) \geq \Lambda_{1,1} e^{-\frac{\lambda_{1,1}}{1+\alpha \lambda_{1,1}} t} > 0 \quad \text{for all } t \geq 0.$$

Now we prove continuity. Since $0 < e^{-\frac{\lambda_{k,j}}{1+\alpha \lambda_{k,j}} t} \leq 1$ for all $t \geq 0$, we have

$$\left| \Lambda_{k,j} e^{-\frac{\lambda_{k,j}}{1+\alpha \lambda_{k,j}} t} \right| \leq |\Lambda_{k,j}|.$$

From Lemma 5, $\rho_{k,j} \le C/k$. The Fourier coefficients $h_{k,j}$ satisfy $\sum_{j,k} h_{k,j}^2 < \infty$ (since $h \in L_2(0,1)$). Therefore,

$$|\Lambda_{k,j}| = \frac{|h_{k,j}|\rho_{k,j}}{1+\alpha\lambda_{k,j}} \le C\frac{|h_{k,j}|}{k^3}.$$

Now we estimate the sum:

$$\begin{aligned}\sum_{j=1}^{2}\sum_{k=1}^{\infty}|\Lambda_{k,j}| &\le C\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{|h_{k,j}|}{k^3}\\ &\le C\left(\sum_{j=1}^{2}\sum_{k=1}^{\infty}h_{k,j}^2\right)^{1/2}\left(\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{1}{k^6}\right)^{1/2} < \infty,\end{aligned}$$

where we used the Cauchy-Schwarz inequality. Since $\sum 1/k^6 < \infty$ and $\sum h_{k,j}^2 = \|h\|_{L_2}^2 < \infty$, the series converges.

By the Weierstrass M-test with $M_{k,j} = |\Lambda_{k,j}|$, the series

$$K(t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty}\Lambda_{k,j}e^{-\frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}}t}$$

converges uniformly on $[0,\infty)$. Each term is continuous, therefore $K(t)$ is continuous on $[0,\infty)$ as a uniform limit of continuous functions. $\square$

We solve (11) by the Laplace transform. Recall that the Laplace transform of μ is

$$\widetilde{\mu}(p) = \int_0^{\infty} e^{-pt}\mu(t)\,dt, \quad \Re p > 0.$$

Applying the transform to (11) and using the convolution theorem gives

$$\widetilde{f}(p) = \widetilde{K}(p)\widetilde{\mu}(p),$$

so that

$$\widetilde{\mu}(p) = \frac{\widetilde{f}(p)}{\widetilde{K}(p)}, \qquad p = \xi + i\tau,\ \xi > 0,\ \tau \in \mathbb{R}.$$

The inverse Laplace transform yields

$$\mu(t) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}\frac{\widetilde{f}(p)}{\widetilde{K}(p)}e^{pt}dp = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{f}(\xi+i\tau)}{\widetilde{K}(\xi+i\tau)}e^{(\xi+i\tau)t}d\tau. \tag{12}$$

Lemma 7 *For $\xi > 0$ and $\tau \in \mathbb{R}$, the following inequality holds:*

$$|\widetilde{K}(\xi+i\tau)| \ge \frac{C_\xi}{\sqrt{1+\tau^2}},$$

where $C_\xi > 0$ depends only on ξ.

Proof of Lemma 7. Set $\sigma_{k,j} = \frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}}$. Then from the definition of $K(t)$,

$$\widetilde{K}(p) = \int_0^{\infty}K(t)e^{-pt}dt = \sum_{i=1}^{2}\sum_{k=1}^{\infty}\frac{\Lambda_{k,j}}{p+\sigma_{k,j}}.$$

For $p = \xi + i\tau$,

$$\widetilde{K}(\xi + i\tau) = \sum_{j,k} \frac{\Lambda_{k,j}(\xi + \sigma_{k,j})}{(\xi + \sigma_{k,j})^2 + \tau^2} - i\tau \sum_{j,k} \frac{\Lambda_{k,j}}{(\xi + \sigma_{k,j})^2 + \tau^2}$$
$$= \Re\widetilde{K} + i\,\Im\widetilde{K},$$

where

$$\Re\widetilde{K} = \sum_{j,k} \frac{\Lambda_{k,j}(\xi + \sigma_{k,j})}{(\xi + \sigma_{k,j})^2 + \tau^2}, \qquad \Im\widetilde{K} = -\tau \sum_{j,k} \frac{\Lambda_{k,j}}{(\xi + \sigma_{k,j})^2 + \tau^2}.$$

Since $(\xi + \sigma_{k,j})^2 + \tau^2 \leq ((\xi + \sigma_{k,j})^2 + 1)(1 + \tau^2)$, we have

$$\frac{1}{(\xi + \sigma_{k,j})^2 + \tau^2} \geq \frac{1}{1 + \tau^2} \cdot \frac{1}{(\xi + \sigma_{k,j})^2 + 1}.$$

Therefore

$$|\Re\widetilde{K}| \geq \frac{1}{1 + \tau^2} \sum_{j,k} \frac{\Lambda_{k,j}(\xi + \sigma_{k,j})}{(\xi + \sigma_{k,j})^2 + 1} = \frac{C_{1,\xi}}{1 + \tau^2},$$

$$|\Im\widetilde{K}| \geq \frac{|\tau|}{1 + \tau^2} \sum_{j,k} \frac{\Lambda_{k,j}}{(\xi + \sigma_{k,j})^2 + 1} = \frac{C_{2,\xi}|\tau|}{1 + \tau^2},$$

where $C_{1,\xi}, C_{2,\xi} > 0$ are finite because $\sum_{j,k} \Lambda_{k,j} < \infty$ (this follows from the Cauchy-Schwarz inequality and the fact that $h \in L_2(0,1)$). Consequently, we obtain

$$|\widetilde{K}(\xi + i\tau)|^2 \geq \frac{C_{1,\xi}^2 + C_{2,\xi}^2\tau^2}{(1 + \tau^2)^2} \geq \frac{\min(C_{1,\xi}^2, C_{2,\xi}^2)}{1 + \tau^2},$$

which proves the lemma. □

Taking the limit $\xi \to 0$ in Lemma 7, we obtain

$$|\widetilde{K}(i\tau)| \geq \frac{C_0}{\sqrt{1 + \tau^2}}, \qquad C_0 := \min(C_{1,0}, C_{2,0}) > 0.$$

Then from (12) we have

$$\mu(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\widetilde{f}(i\tau)}{\widetilde{K}(i\tau)} e^{i\tau t} d\tau. \tag{13}$$

Lemma 8 *[18] Assume that $f \in W(M)$. Then*

$$\int_{-\infty}^{+\infty} |\widetilde{f}(i\tau)|\sqrt{1 + \tau^2}\, d\tau \leq C_* \|f\|_{W_2^2(\mathbb{R}_+)},$$

where $C_* > 0$ is a constant.

Proof of Theorem 2. We first show that $\mu \in L^2(\mathbb{R}_+)$. From Lemma 7 with $\xi \to 0$,

$$|\widetilde{\mu}(\tau)| = \left|\frac{\widetilde{f}(i\tau)}{\widetilde{K}(i\tau)}\right| \leq \frac{1}{C_0}|\widetilde{f}(i\tau)|\sqrt{1 + \tau^2}.$$

Hence,

$$\int_{-\infty}^{+\infty} |\widetilde{\mu}(\tau)|^2 d\tau \leq \frac{1}{C_0^2} \int_{-\infty}^{+\infty} |\widetilde{f}(i\tau)|^2 (1 + \tau^2) d\tau < \infty,$$

since $f \in W_2^2(\mathbb{R}_+)$. Thus $\widetilde{\mu} \in L^2(\mathbb{R})$, and by the Plancherel theorem, $\mu \in L^2(\mathbb{R})$. Because $\mu(t) = 0$ for $t < 0$, we conclude $\mu \in L^2(\mathbb{R}_+)$.

Next, from (13) and Lemmas 7 and 8,

$$\begin{aligned}|\mu(t)| &\leq \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|\widetilde{f}(i\tau)|}{|\widetilde{K}(i\tau)|}d\tau\\ &\leq \frac{1}{2\pi C_0}\int_{-\infty}^{+\infty}|\widetilde{f}(i\tau)|\sqrt{1+\tau^2}\,d\tau\\ &\leq \frac{C_*}{2\pi C_0}\|f\|_{W_2^2(\mathbb{R}_+)} \leq \frac{C_* M}{2\pi C_0}.\end{aligned}$$

Choosing $M = \dfrac{2\pi C_0}{C_*}$, we obtain $|\mu(t)| \leq 1$ for every $f \in W(M)$. This completes the proof. □

Example. We now present a concrete illustration of the control construction proposed above. Recall that, according to the integral condition (7), $f(t)$ represents the prescribed trajectory of the weighted average temperature

$$\int_0^1 \rho(x)\,u(x,t)\,dx = f(t), \qquad t \geq 0,$$

where $\rho \in U$ is a given weight function. In this example, we consider the desired trajectory

$$f(t) = \begin{cases} 0, & t \leq 0,\\ M_0 t^2 e^{-t}, & t > 0,\end{cases}$$

where $M_0 > 0$ is a constant to be determined. It is straightforward to verify that $f(0) = 0$ and $f \in W(M)$ for an appropriate choice of M.

Recall that the kernel of the Volterra integral equation (11) is given by

$$K(t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty}\Lambda_{k,j}e^{-\sigma_{k,j}t}, \qquad \sigma_{k,j} = \frac{\lambda_{k,j}}{1+\alpha\lambda_{k,j}}.$$

Since $\alpha > 0$ and $\lambda_{k,j} > 0$, it follows that $0 < \sigma_{k,j} < 1$ for all k, j. For the purpose of this illustrative example, we retain only the leading term of the series, namely

$$K(t) \approx \Lambda_{1,1}e^{-\sigma_{1,1}t},$$

where $\Lambda_{1,1} > 0$ and $0 < \sigma_{1,1} < 1$. This simplification is introduced solely to demonstrate the construction explicitly and is not used in the proof of Theorem 2.

Under this approximation, the Volterra integral equation (11) reduces to

$$\Lambda_{1,1}\int_0^t e^{-\sigma_{1,1}(t-s)}\mu(s)\,ds = M_0 t^2 e^{-t}, \qquad t > 0.$$

Applying the Laplace transform and using the convolution theorem, we obtain

$$\widetilde{\mu}(p) = \frac{2M_0}{\Lambda_{1,1}}\cdot\frac{p+\sigma_{1,1}}{(p+1)^3}.$$

Taking the inverse Laplace transform yields the explicit control function

$$\mu(t) = \frac{M_0}{\Lambda_{1,1}}\,e^{-t}\Big[2t - (1-\sigma_{1,1})t^2\Big].$$

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ЛОКАЛДЫ ЕМЕС ШЕКАРАЛЫҚ ШАРТЫ БАР БІР ӨЛШЕМДІ ПСЕВДОПАРАБОЛДЫҚ ТЕҢДЕУ ҮШІН БАСҚАРУ МӘСЕЛЕСІ ТУРАЛЫ

Аңдатпа

Бұл жұмыста Бицадзе-Самарский типіндегі локалды емес шекаралық шарты бар бір өлшемді псевдопараболалық теңдеу үшін басқару мәселесі зерттеледі. Бақылау функциясы теңдеудің оң жағында таратылған көз түрінде кіреді, мақсат жүйені нөлдік бастапқы күйден басқару арқылы салмақталған орташа температураның берілген траектория бойынша өзгеруін қамтамасыз ету болып табылады. Теңдеу жылу өткізгіштіктегі жады немесе тұтқыр серпімділік әсерлерін ескеретін үшінші ретті аралас туынды мүшені қамтиды, бұл әсіресе полимерлер, биологиялық ұлпалар және түйіршікті материалдар сияқты ішкі құрылымы бар материалдар үшін маңызды. Спектрлік жіктеу және биортогоналды базис көмегімен есеп бірінші текті Вольтерра интегралдық теңдеуіне келтіріледі. Бұл интегралдық теңдеудің ядросының оң және үзіліссіз екендігі дәлелденген. Салмақ функциясына және басқарудың кеңістіктік таралуына сәйкес болжамдарда нормасы жеткілікті аз болатын қажетті траекториялар класы үшін рұқсат етілген басқарудың бар екендігі дәлелденген. Интегралдық теңдеуді шешу және басқару функциясына бағалау алу үшін Лаплас түрлендіру әдісі қолданылады. Нәтижелер локалды емес шекаралық шарт пен жады параметрінің басқарылу қабілетіне әсерін сандық түрде сипаттайды.

Түйін сөздер: псевдопараболдық теңдеу, шекаралық есеп, Вольтерра интегралдық теңдеуі, рұқсат етілген басқару, жіңішке өзек.

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О ЗАДАЧЕ УПРАВЛЕНИЯ ДЛЯ ОДНОМЕРНОГО ПСЕВДОПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ С НЕЛОКАЛЬНЫМ ГРАНИЧНЫМ УСЛОВИЕМ

Аннотация

В настоящей статье исследуется задача управления для одномерного псевдопараболического уравнения с нелокальным граничным условием типа Бицадзе-Самарского. Управляющая функция входит в правую часть уравнения в виде распределенного источника, и цель состоит в том, чтобы перевести систему из нулевого начального состояния так, чтобы средневзвешенная температура следовала заданной траектории. Уравнение содержит член третьего порядка со смешанной производной, который учитывает эффекты памяти или вязкоупругости в теплопроводности, что особенно важно для материалов с внутренней структурой, таких как полимеры, биологические ткани и гранулированные среды. С помощью спектрального разложения и биортогонального базиса задача сводится к интегральному уравнению Вольтерра первого рода. Доказано, что ядро этого интегрального уравнения положительно и непрерывно. При соответствующих предположениях о весовой функции и пространственном распределении управления доказано существование допустимого управления для класса желаемых траекторий с достаточно малой нормой. Для решения интегрального уравнения и получения оценок управляющей функции используется метод преобразования Лапласа. Результаты количественно характеризуют влияние нелокального граничного условия и параметра памяти на управляемость системы.

Ключевые слова: псевдопараболическое уравнение, краевая задача, интегральное уравнение Вольтерра, допустимое управление, тонкий стержень.