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[1,2]**Koilyshov U.,**
Cand.Phys.-Math.Sc., Associate Professor, ORCID ID: 0000-0003-1752-7848,
e-mail: koylyshov@math.kz
[1,2*]**Zhapparova S.,**
PhD student, ORCID ID: 0009-0007-4580-0983,
*e-mail: zhapparova.saule@gmail.com
[3]**Nietbayev A.,**
Cand.Phys.-Math.Sc., Associate Professor, ORCID ID: 0009-0006-5677-5464,
e-mail: a.niet57@mail.ru

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[2]Al-Farabi Kazakh National University, Almaty, Kazakhstan
[3]Dulaty University, Taraz, Kazakhstan

TWO-PHASE HEAT CONDUCTION PROBLEMS WITH STURM-TYPE BOUNDARY CONDITIONS AND WITH A FRACTIONAL TIME DERIVATIVE

Abstract

The paper investigates an initial-boundary value problem of the Sturm type for the heat equation with a piecewise-constant coefficient and a Caputo derivative of order $0 < \alpha < 1$ with respect to time. The domain under consideration is an interval $(0, l)$ containing a strictly internal point of discontinuity $x = x_0$. From a physical point of view, this problem models the process of temperature field propagation in a thin rod of length l. The rod is composite and consists of two sections – $(0, x_0)$ and (x_0, l), – possessing different thermophysical characteristics. At the contact point of the two media $x = x_0$, perfect contact conditions are prescribed, implying continuity of temperature and continuity of heat flux when passing from one medium to another. The main goal of the work is to substantiate the solution of the posed initial-boundary value problem by the method of separation of variables (Fourier method). Application of this method leads to the necessity of investigating the corresponding spectral problem for an ordinary differential operator with a discontinuous coefficient at the highest derivative.

Keywords: heat equation, fractional derivatives, discontinuous coefficients, eigenvalues, eigenfunctions, method of separation of variables.

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Introduction

The theory of fractional order differential equations has received powerful development in recent decades, which is due to the broad possibilities of its applications in describing processes with memory and heredity effects, as well as phenomena of anomalous diffusion in fractal media. The fundamental bases of the theory, as well as methods for solving such equations, are detailed in monographs and in the classical works of Gorenflo and Mainardi [1], where the connection between fractional calculus and problems of continuum mechanics is discussed.

Questions of the well-posedness of initial-boundary value problems for the diffusion equation with a fractional time derivative (in the Caputo or Riemann-Liouville sense) have been investigated by many authors. In the works of Wyss [2] and Mainardi [3], fundamental solutions were constructed and their properties studied.

For the investigation of boundary value problems in bounded domains, works devoted to spectral methods and maximum principles are of great importance. In particular, Luchko [4] substantiated the maximum principle for the generalized fractional diffusion equation, which is a key factor for proving the uniqueness of solutions. The application of the method of separation of variables for fractional order equations in bounded domains is considered in detail in the article by Agrawal [5], and the fundamental theory of boundary value problems for such equations is developed in the works of Pskhu [6].

On the other hand, heat conduction problems in inhomogeneous (layered) media lead to differential equations with discontinuous coefficients. The classical formulation of such problems with conjugation conditions (perfect contact conditions) was formulated by A.A. Samarskii [7]. The presence of discontinuities in coefficients significantly affects the spectral properties of the corresponding differential operators. Questions of the asymptotic behavior of eigenvalues of Sturm-Liouville operators with discontinuous coefficients were investigated in the works of Hald [8].

A particular difficulty in such problems is the question of the basis property of the system of eigenfunctions. It is known that for non-self-adjoint operators arising in problems with non-local conditions or discontinuous coefficients, the system of eigenfunctions may not form a basis in the classical sense, but may form a Riesz basis. Conditions for the basis property of systems of eigenfunctions for problems with conjugation conditions were studied in the works of V.A. Ilyin [9], as well as N.Y. Kapustin and E.I. Moiseev [10]. Recent results regarding the Riesz basis property for operators with discontinuous coefficients and their application to two-phase heat conduction problems were obtained in works [11–14].

However, despite a significant number of works in the fields of fractional calculus and equations with discontinuous coefficients separately, problems combining both these aspects – a fractional time derivative and discontinuous coefficients with respect to the spatial variable – have been insufficiently studied. In particular, questions regarding the substantiation of the Fourier method and the proof of the Riesz basis property of eigenfunctions for such "hybrid" problems require separate consideration, to which the present work is dedicated.

This research stands out because it considers a specific boundary value problem combining discontinuous coefficients with fractional differentiation with respect to time.

In our work, we investigate an initial-boundary value problem for the heat equation with a piecewise-constant coefficient and a Caputo derivative of order $0 < \alpha < 1$ with respect to time. The domain under consideration is an interval $(0, l)$, containing a strictly internal point of discontinuity $x = x_0$.

From a physical point of view, this problem models the process of temperature field propagation in a thin rod of length l. The rod is composite and consists of two sections – $(0, x_0)$ and (x_0, l), – possessing different thermophysical characteristics.

At the contact point of the two media $x = x_0$, perfect contact conditions are prescribed, implying continuity of temperature and continuity of heat flux when passing from one medium to another. These conditions are natural and follow directly from the equation and the presence of discontinuities in the thermophysical characteristics (coefficients of thermal conductivity, density, and heat capacity) [7].

The main goal of the work is to substantiate the solution of the posed initial-boundary value problem by the method of separation of variables (Fourier method). Application of this method leads to the necessity of investigating the corresponding spectral problem for an ordinary differential operator with a discontinuous coefficient at the highest derivative.

Materials and methods

An initial-boundary value problem for the heat equation with a piecewise constant coefficient and with a fractional time derivative is considered

$$\mathcal{L}u \equiv \begin{Bmatrix} t^{-\beta} D_t^{\alpha} u(x,t) - a_1^2 u_{xx}(x,t), & 0 < x < x_0 \\ t^{-\beta} D_t^{\alpha} u(x,t) - a_2^2 u_{xx}(x,t), & x_0 < x < l \end{Bmatrix} = 0, \tag{1}$$

in the domain

$$\Omega = \bigcup \Omega_i,\ \Omega_1 = \{(x,t): 0 < x < x_0,\ \ 0 < t < T\},$$
$$\Omega_2 = \{(x,t): x_0 < x < l,\ \ 0 < t < T\},\ (i = 1,2),$$

with the initial condition

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \tag{2}$$

and boundary conditions

$$\begin{cases} a_{11} u_x(0,t) + a_{12} u(0,t) = 0, \\ a_{21} u_x(l,t) + a_{22} u(l,t) = 0, \end{cases} \quad 0 \le t \le T, \tag{3}$$

and with conjugation conditions

$$\begin{cases} u(x_0 - 0, t) = u(x_0 + 0, t), \\ k_1 u_x(x_0 - 0, t) = k_2 u_x(x_0 + 0, t), \end{cases} \quad 0 \le t \le T. \tag{4}$$

Here D_t^{α} denotes the Caputo fractional derivative of order $\alpha \in (0,1)$, $\beta > 0$,

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau.$$

The coefficients a_{ij}, $(i,j = 1,2)$ are real numbers. In addition

$$|a_{11}| + |a_{12}| > 0,\ |a_{21}| + |a_{22}| > 0,$$
$$a_i^2 = \frac{k_i}{c_i \rho_i},\ k_i > 0,\ (i = 1,2),$$

where k_i, $\left(\frac{w}{m \cdot K}\right)$ is the thermal conductivity coefficient, c_i, $\left(\frac{j}{K \cdot kg}\right)$ is the specific heat capacity, ρ_i, $\left(\frac{kg}{m^3}\right)$ is the density, a_i^2, $\left(\frac{m^2}{s}\right)$ is the thermal diffusivity coefficient.
The $\varphi(x)$ a twice continuously differentiable function that satisfies the boundary conditions (3) and conjugation conditions (4).

The solution of problem (1)-(4) is sought in the form $u(x,t) = X(x) \cdot T(t) \neq 0$.
Substituting into equation (1) and conditions (2)-(4), and separating the variables, we obtain the following spectral problem

$$\mathcal{L}X(x) = \begin{Bmatrix} -a_1^2 X''(x), & 0 < x < x_0 \\ -a_2^2 X''(x), & x_0 < x < l \end{Bmatrix} = \lambda X(x), \tag{5}$$

$$\begin{cases} a_{11} X'(0) + a_{12} X(0) = 0, \\ a_{21} X'(l) + a_{22} X(l) = 0, \end{cases} \tag{6}$$

$$\begin{cases} X(x_0 - 0) = X(x_0 + 0), \\ k_1 X'(x_0 - 0) = k_2 X'(x_0 + 0). \end{cases} \tag{7}$$

The function $T(t)$ is a solution to the equation

$$t^{-\beta} D_t^{\alpha} T(t) + \lambda T(t) = 0. \tag{8}$$

The general solution of equation (5) has the form

$$\begin{cases} X(x) = c_1 \cos(\mu_1 x) + c_2 \sin(\mu_1 x), & 0 < x < x_0, \\ X(x) = c_3 \cos(\mu_2 (l - x)) + c_4 \sin(\mu_2 (l - x)), & x_0 < x < l, \end{cases} \quad (9)$$

where

$$\mu_i = \frac{\sqrt{\lambda}}{a_i}, (i = 1,2). \quad (10)$$

Substituting the general solution (9) into the boundary conditions (6) and conjugation conditions (7), we get

$$\begin{cases} c_1 a_{12} + c_2 a_{11} \mu_1 = 0, \\ c_3 a_{22} - c_4 a_{21} \mu_2 = 0, \\ c_1 \cos(\mu_1 x_0) + c_2 \sin(\mu_1 x_0) - c_3 \cos(\mu_2 (l - x_0)) - c_4 \sin(\mu_2 (l - x_0)) = 0, \\ -c_1 k_1 \mu_1 \sin(\mu_1 x_0) + c_2 k_1 \mu_1 \cos(\mu_1 x_0) - c_3 k_2 \mu_2 \sin(\mu_2 (l - x_0)) + c_4 k_2 \mu_2 \cos(\mu_2 (l - x_0)) = 0. \end{cases}$$

Taking into account formula (10), we find the characteristic determinant of the system:

$$\Delta(\lambda) = a_{11} a_{21} \lambda \left(\frac{k_1}{a_1} \sin(\mu_1 x_0) \cos(\mu_2 (l - x_0)) + \frac{k_2}{a_2} \cos(\mu_1 x_0) \sin(\mu_2 (l - x_0)) \right) -$$
$$-\sqrt{\lambda} \left(\frac{a_{12} a_{21} a_1 k_2}{a_2} - \frac{a_{11} a_{22} a_2 k_1}{a_1} \right) \sin(\mu_1 x_0) \sin(\mu_2 (l - x_0)) +$$
$$+\sqrt{\lambda} (a_{11} a_{22} k_2 - a_{21} a_{12} k_1) \cos(\mu_1 x_0) \cos(\mu_2 (l - x_0)) +$$
$$+a_{12} a_{22} (a_1 k_2 \sin(\mu_1 x_0) \cos(\mu_2 (l - x_0)) + a_2 k_1 \cos(\mu_1 x_0) \sin(\mu_2 (l - x_0))) = 0. \quad (11)$$

The roots of equation (11) will be the eigenvalues of problem (5)-(7). It is not possible to find the eigenvalues of λ_n explicitly, but it is possible to construct asymptotics. It will be shown that for $\lambda = 0$, the spectral problem (5)-(7) has only a zero solution.

Next, the eigenfunctions of the spectral problem (5)-(7) are constructed. They look as follows:

$$X_n(x) = C_n \begin{cases} \Phi_2(x_0, \lambda_n) \Phi_1(x, \lambda_n), & 0 < x < x_0, \\ \Phi_1(x_0, \lambda_n) \Phi_2(x, \lambda_n), & x_0 < x < l, \end{cases} \quad (12)$$

where

$$\Phi_1(x, \lambda_n) = \cos\left(\frac{\sqrt{\lambda_n}}{a_1} x\right) - \frac{a_{12} a_1}{a_{11} \sqrt{\lambda_n}} \sin\left(\frac{\sqrt{\lambda_n}}{a_1} x\right), \quad (13)$$

$$\Phi_2(x, \lambda_n) = \cos\left(\frac{\sqrt{\lambda_n}}{a_2} (l - x)\right) + \frac{a_{22} a_2}{a_{21} \sqrt{\lambda_n}} \sin\left(\frac{\sqrt{\lambda_n}}{a_2} (l - x)\right), \quad (14)$$

where λ_n are roots of equation (11).

It is easy to show that eigenfunctions (12) satisfy equation (5), boundary conditions (6) and conjugation conditions (7).

The following Lemma is true.

Lemma 1. Spectral problem (5)-(7) is not self-adjoint.

An adjoint problem to problem (5)-(7) has the following form:

$$\mathcal{L} Y(x) = \begin{Bmatrix} -a_1^2 Y''(x), & 0 < x < x_0 \\ -a_2^2 Y''(x), & x_0 < x < l \end{Bmatrix} = \lambda Y(x), \quad (15)$$

$$\begin{cases} a_{11} Y'(0) + a_{12} Y(0) = 0, \\ a_{21} Y'(l) + a_{22} Y(l) = 0, \end{cases} \quad (16)$$

$$\begin{cases} \frac{a_1^2}{k_1} Y(x_0 - 0) = \frac{a_2^2}{k_2} Y(x_0 + 0), \\ a_1^2 Y'(x_0 - 0) = a_2^2 Y'(x_0 + 0). \end{cases} \quad (17)$$

Proof. The lemma can be proved by direct calculation.

Therefore, it cannot be asserted that the eigenfunctions $X_n(x)$ form the basis in L_2. It is easy to show that they are not orthogonal.

In a similar way, one can find the eigenvalues and construct the eigenfunctions of the adjoint problem (15)-(17). Direct calculation can show that the eigenvalues of the spectral problems (5)-(7) and (15)-(17) coincide. The eigenfunctions of problem (15)-(17) have the form:

$$Y_n(x) = C_n \begin{cases} \frac{k_1}{a_1^2} \Phi_2(x_0, \lambda_n) \Phi_1(x, \lambda_n), & 0 < x < x_0, \\ \frac{k_2}{a_2^2} \Phi_1(x_0, \lambda_n) \Phi_2(x, \lambda_n), & x_0 < x < l, \end{cases} \tag{18}$$

where λ_n are roots of equation (11), functions $\Phi_1(x, \lambda_n)$ $\Phi_2(x, \lambda_n)$ are determined by formulas (13)-(14).

The following Lemma holds.

Lemma 2. The systems of eigenfunctions (12) and (18) are biorthogonal on the interval $(0, l)$, so that for any numbers i and j the relation holds

$$(X_i, Y_j) = \int_0^l X_i(x) Y_j(x) dx = \delta_j^i,$$

where δ_j^i is the Kronecker symbol.

The lemma can be proved by direct calculation, taking into account that the eigenvalues of spectral problems (5)-(7) and (15)-(17) coincide, and the eigenfunctions (12) and (18) differ by a constant factor.

Next, a self-adjoint problem is constructed.

Lemma 3. The following spectral problem is self-adjoint.

$$\mathcal{L}Z(x) = \begin{Bmatrix} -a_1^2 Z''(x), & 0 < x < x_0 \\ -a_2^2 Z''(x), & x_0 < x < l \end{Bmatrix} = \lambda Z(x), \tag{19}$$

$$\begin{cases} a_{11} Z'(0) + a_{12} Z(0) = 0, \\ a_{21} Z'(l) + a_{22} Z(l) = 0, \end{cases} \tag{20}$$

$$\begin{cases} \frac{a_1}{\sqrt{k_1}} Z(x_0 - 0) = \frac{a_2}{\sqrt{k_2}} Z(x_0 + 0), \\ a_1 \sqrt{k_1} Z'(x_0 - 0) = a_2 \sqrt{k_2} Z'(x_0 + 0). \end{cases} \tag{21}$$

Proof. Considering the following formula

$$-Z''(x)V(x) = \big(V'(x)Z(x) - V(x)Z'(x)\big)' - V''(x)Z(x),$$

we get

$$\int_0^l V(x)\mathcal{L}Z(x)dx = -\int_0^{x_0} V(x) a_1^2 Z''(x) dx - \int_{x_0}^l V(x) a_2^2 Z''(x) dx =$$
$$= a_1^2 Z(x_0 - 0) V'(x_0 - 0) - a_1^2 Z(0) V'(0) - a_1^2 Z'(x_0 - 0) V(x_0 - 0) +$$
$$+ a_1^2 Z'(0) V(0) + a_2^2 Z(l) V'(l) - a_2^2 Z(x_0 + 0) V'(x_0 + 0) - a_2^2 Z'(l) V(l) +$$
$$+ a_2^2 Z'(x_0 + 0) V(x_0 + 0) + \int_0^l Z(x) \mathcal{L} V(x) dx.$$

By taking into account the boundary conditions

$$\int_0^l V(x)\mathcal{L}Z(x)dx = a_1\sqrt{k_1} V'(x_0 - 0) \frac{a_1}{\sqrt{k_1}} Z(x_0 - 0) - a_1 \sqrt{k_1} Z'(x_0 - 0) \frac{a_1}{\sqrt{k_1}} V(x_0 - 0) +$$
$$+ a_2 \sqrt{k_2} V'(x_0 + 0) \frac{a_2}{\sqrt{k_2}} Z(x_0 + 0) + a_2 \sqrt{k_2} Z'(x_0 + 0) \frac{a_2}{\sqrt{k_2}} V(x_0 + 0) + \int_0^l Z(x) \mathcal{L} V(x) dx.$$

Further, in a similar way, one can calculate the eigenvalues of problem (19)-(21), they will be the roots of equation (11), that is, they coincide with the eigenvalues of problems (5)-(7) and (15)-(17).

The eigenvalues of the function are written as:

$$Z_n(x) = C_n \begin{cases} \frac{\sqrt{k_1}}{a_1^2} \Phi_2(x_0, \lambda_n) \Phi_1(x, \lambda_n), & 0 < x < x_0, \\ \frac{\sqrt{k_2}}{a_2^2} \Phi_1(x_0, \lambda_n) \Phi_2(x, \lambda_n), & x_0 < x < l, \end{cases} \tag{22}$$

where λ_n are roots of equation (11).

Lemma 4. The system of eigenfunctions (22) forms an orthonormal system.

The proof follows from the general theory of self-adjoint problems. C_n can be found from the normalization condition

$$C_n = \left(\frac{k_1}{a_1^2}\Phi_2^2(x_0,\lambda_n)\int_0^{x_0}\Phi_1^2(x,\lambda_n)dx + \frac{k_2}{a_2^2}\Phi_1^2(x_0,\lambda_n)\int_{x_0}^{l}\Phi_2^2(x,\lambda_n)dx\right)^{-\frac{1}{2}}.$$

Lemma 5. The eigenfunctions $X_n(x)$ of the spectral problem (5)-(7) defined by formula (12) form a Riesz basis.

Proof. In the previous lemma it has been proved that the system of eigenfunctions $\{Z_n(x)\}$ forms an orthonormal basis in $L_2(0,l)$. From formulas (12) and (20) it is clear that the eigenfunctions of problems (5)-(7) and (17)-(19) are related by the following equality:

$$Z_n(x) = \alpha(x)X_n(x), \text{ where } \alpha(x) = \begin{cases} \frac{\sqrt{k_1}}{a_1}, & 0 < x < x_0, \\ \frac{\sqrt{k_2}}{a_2}, & x_0 < x < l. \end{cases}$$

$AX(x) = \alpha(x)X(x)$, $A: L_2(0,l) \to L_2(0,l)$ is a bounded operator and there exists A^{-1}, which is also a bounded operator. From the definition of the Riesz basis it follows that the system of eigenfunctions $\{X_n(x)\}$ forms a Riesz basis.

From equation (8):

$$t^{-\beta}D_t^{\alpha}T(t) + \lambda T(t) = 0,$$

we find the function

$$T_n(t) = \varphi_n E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_n t^{\alpha+\beta}\right),$$

where $E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(z)$ – generalized Mittag-Leffler function. It is determined by the following formula:

$$E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(z) = \sum_{k=0}^{\infty} c_k z^k,\ c_0 = 1,\ c_k = \prod_{j=0}^{k-1}\frac{\Gamma(j(\alpha+\beta)+\beta+1)}{\Gamma(j(\alpha+\beta)+\alpha+\beta+1)},$$

$\Gamma(z)$ – gamma function.

The following theorem holds.

Results and discussion

Theorem. Let $\varphi(x)$ be a twice continuously differentiable function satisfying the conditions

$$a_{11}\varphi'(0) + a_{12}\varphi(0) = a_{21}\varphi'(l) + a_{22}\varphi(l) = 0,$$

$$\varphi(x_0 - 0) = \varphi(x_0 + 0),\ k_1\varphi'(x_0 - 0) = k_2\varphi'(x_0 + 0). \quad (23)$$

Then the function

$$u(x,t) = \sum_{n=1}^{\infty}\varphi_n X_n(x)\, E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_n t^{\alpha+\beta}\right), \quad (24)$$

where the coefficients φ_n are determined by the formula

$$\varphi_n = \int_0^l \varphi(x)Y_n(x)dx, \quad (25)$$

is a unique solution to problem (1)-(4).

Proof. First, we prove the existence of a solution (24). Since $\{X_n(x)\}$ the eigenfunctions and $\lambda_n -$ eigenvalues of problem (5)-(7), it is easy to verify that the function determined by formula (22) satisfies the equation, initial condition, boundary conditions and conjugation conditions of problem (1)-(4). The series (24) is the sum of functions

$$u_n(x,t) = \varphi_n X_n(x) E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_n t^{\alpha+\beta}\right). \quad (26)$$

Let us show that for $t \geq \varepsilon > 0$ (ε - any positive number) the series

$$\sum_{n=1}^{\infty} u_n(x,t),\ \sum_{n=1}^{\infty} t^{-\beta}D_t^{\alpha}u_n,\ \sum_{n=1}^{\infty}\frac{\partial^2 u_n}{\partial x^2},$$

converges uniformly. Obviously, $|\varphi| \leq M_1$, then from formula (25) it follows that $|\varphi_n| \leq M_2$. Let us show that the series (24) converges uniformly, for this, we estimate the n-th term as follows:

$$\sup_{(x,t)\in\bar{\Omega}}|X_n(x)T_n(t)| \leq M_3 \sup_{0\leq t\leq T}|T_n(t)| = M_3 \sup_{0\leq t\leq T}\left|\varphi_n E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_n t^{\alpha+\beta}\right)\right|.$$

Using the known estimate [15]:

$$\left|E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(z)\right| \le \frac{M_4}{1+|z|},\ \arg(z)=\pi,\ |z|\to\infty,\ M_4=\mathrm{const}>0 \tag{27}$$

we get $\sup_{(x,t)\in\bar\Omega}|X_n(x)T_n(t)| \le \frac{C}{\lambda_n}$. Since the majorizing series $\sum_{n=1}^{\infty}\frac{C}{\lambda_n}$, converges, the series (25) converges uniformly on $\bar\Omega$ to some function $u(x,t)$. Now we will show the possibility of term-by-term differentiation of series (24). To do this, it is sufficient to show the uniform convergence of the series $\sum_{n=1}^{\infty} t^{-\beta}D_t^{\alpha}u_n$, $\sum_{n=1}^{\infty}\frac{\partial^2 u_n}{\partial x^2}$ then the possibility of term-by-term differentiation will be justified. Now let's look at the series $\sum_{n=1}^{\infty}\frac{\partial^2 u_n}{\partial x^2}$. Let us estimate the n-th term of the series.

Using the property $X_n''(x)=\lambda_n X_n(x)$ we obtain

$$\sum_{n=1}^{\infty}\frac{\partial^2 u_n}{\partial x^2}=\sum_{n=1}^{\infty}\frac{\lambda_n}{k_i^2}\varphi_n X_n(x)E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_n t^{\alpha+\beta}\right). \tag{28}$$

Now let us consider the integral (25). Integrating by parts we have

$$\varphi_n=\int_0^l\varphi(x)Y_n(x)dx=\frac{C_nk_1}{a_1^2}\left(\int_0^{x_0}\varphi(x)\Phi_2(x_0,\lambda_n)\left(\cos\left(\frac{\sqrt{\lambda_n}}{a_1}x\right)-\frac{a_{12}a_1}{a_{11}\sqrt{\lambda_n}}\sin\left(\frac{\sqrt{\lambda_n}}{a_1}x\right)\right)dx\right)+$$
$$+\frac{C_nk_2}{a_2^2}\left(\int_{x_0}^{l}\varphi(x)\Phi_1(x_0,\lambda_n)\left(\cos\left(\frac{\sqrt{\lambda_n}}{a_2}(l-x)\right)+\frac{a_{22}a_2}{a_{21}\sqrt{\lambda_n}}\sin\left(\frac{\sqrt{\lambda_n}}{a_2}(l-x)\right)\right)dx\right)=$$
$$=\frac{C_nk_1}{a_1}\Phi_2(x_0,\lambda_n)\left(\frac{\varphi(x_0-0)}{\sqrt{\lambda_n}}\left(\sin\left(\frac{\sqrt{\lambda_n}}{a_1}x_0\right)+\frac{a_{12}a_1}{a_{11}\sqrt{\lambda_n}}\cos\left(\frac{\sqrt{\lambda_n}}{a_1}x_0\right)\right)-\frac{a_{12}a_1}{a_{11}\lambda_n}\varphi(0)\right)-$$
$$-\frac{C_nk_1}{a_1}\left(\int_0^{x_0}\frac{\varphi'(x)}{\sqrt{\lambda_n}}\Phi_2(x_0,\lambda_n)\left(\frac{a_{12}a_1}{a_{11}\sqrt{\lambda_n}}\cos\left(\frac{\sqrt{\lambda_n}}{a_1}x\right)+\sin\left(\frac{\sqrt{\lambda_n}}{a_1}x\right)\right)dx\right)+$$
$$+\frac{C_nk_2}{a_2}\Phi_1(x_0,\lambda_n)\left(-\frac{\varphi(x_0+0)}{\sqrt{\lambda_n}}\left(-\sin\left(\frac{\sqrt{\lambda_n}}{a_2}(l-x_0)\right)+\frac{a_{22}a_2}{a_{21}\sqrt{\lambda_n}}\cos\left(\frac{\sqrt{\lambda_n}}{a_2}(l-x_0)\right)\right)+\frac{a_{22}a_2}{a_{21}\lambda_n}\varphi(l)\right)-$$
$$-\frac{C_nk_2}{a_2}\left(\int_{x_0}^{l}\frac{\varphi'(x)}{\sqrt{\lambda_n}}\Phi_1(x_0,\lambda_n)\left(\frac{a_{22}a_2}{a_{21}\sqrt{\lambda_n}}\cos\left(\frac{\sqrt{\lambda_n}}{a_2}(l-x)\right)-\sin\left(\frac{\sqrt{\lambda_n}}{a_2}(l-x)\right)\right)dx\right).$$

Integrating by parts again, we obtain

$$\varphi_n=\frac{C_nk_1}{a_1}\Phi_2(x_0,\lambda_n)\left(\frac{\varphi(x_0-0)}{\sqrt{\lambda_n}}\left(\sin\left(\frac{\sqrt{\lambda_n}}{a_1}x_0\right)+\frac{a_{12}a_1}{a_{11}\sqrt{\lambda_n}}\cos\left(\frac{\sqrt{\lambda_n}}{a_1}x_0\right)\right)-\frac{a_{12}a_1}{a_{11}\lambda_n}\varphi(0)\right)+$$
$$+\frac{C_nk_1}{a_1}\Phi_2(x_0,\lambda_n)\left(\frac{\varphi'(x_0-0)a_1}{\lambda_n}\Phi_1(x_0,\lambda_n)-\frac{a_1}{\lambda_n}\varphi'(0)-\frac{a_1}{\lambda_n}\int_0^{x_0}\varphi''(x)\Phi_1(x,\lambda_n)dx\right)+$$
$$+\frac{C_nk_2}{a_2}\Phi_1(x_0,\lambda_n)\left(-\frac{\varphi(x_0+0)}{\sqrt{\lambda_n}}\left(-\sin\left(\frac{\sqrt{\lambda_n}}{a_2}(l-x_0)\right)+\frac{a_{22}a_2}{a_{21}\sqrt{\lambda_n}}\cos\left(\frac{\sqrt{\lambda_n}}{a_2}(l-x_0)\right)\right)+\right.$$
$$\left.\frac{a_{22}a_2}{a_{21}\sqrt{\lambda_n}}\varphi(l)\right)-$$
$$-\frac{C_nk_2}{a_2}\Phi_1(x_0,\lambda_n)\left(\frac{\varphi'(x_0+0)a_2}{\lambda_n}\Phi_2(x_0,\lambda_n)-\frac{a_2}{\lambda_n}\varphi'(l)+\frac{a_2}{\lambda_n}\int_{x_0}^{l}\varphi''(x)\Phi_2(x,\lambda_n)dx\right).$$

Considering that λ_n is the root of the characteristic equation (11), and the function $\varphi(x)$ satisfies conditions (23), we have

$$\varphi_n=-\frac{C_nk_1}{\lambda_n}\Phi_2(x_0,\lambda_n)\int_0^{x_0}\varphi''(x)\Phi_1(x,\lambda_n)dx-$$
$$-\frac{C_nk_2}{\lambda_n}\Phi_1(x_0,\lambda_n)\int_{x_0}^{l}\varphi''(x)\Phi_2(x,\lambda_n)dx=\frac{K}{\lambda_n}\int_0^l\varphi''(x)Z_n(x)dx, \tag{29}$$

where

$$K=\begin{cases}-a_1\sqrt{k_1}, & 0<x<x_0,\\ -a_2\sqrt{k_2}, & x_0<x<l,\end{cases}$$

$Z_n(x)$ is determined by the formula (22).

From formula (29) implies

$$|\varphi_n| \leq K_1 \frac{|A_n|}{n^2}, \quad K_1 = \max\left(-a_1\sqrt{k_1}, -a_2\sqrt{k_2}\right),$$

where $A_n = \int_0^l \varphi''(x) Z_n(x)dx$ - Fourier coefficients of functions $\varphi''(x)$ orthonormalized on a segment $[0, l]$ of eigenfunctions $\{Z_n(x)\}$ determined by formula (22). Then, based on Lemma 3 and Bessel's inequality $\sum_{n=1}^{\infty} A_n^2 \leq \|\varphi''\|^2$, from formulas (27)-(29), we obtain $\sum_{n=1}^{\infty} \left|\frac{\partial^2 u_n}{\partial x^2}\right| = \sum_{n=1}^{\infty} \frac{M_5}{1+|\lambda_n t^{\alpha+\beta}|}$. Thus, the majorizing series converges absolutely, which means that the series (28) converges uniformly at $t \geq \varepsilon > 0$. Uniform convergence of the series $\sum_{n=1}^{\infty} t^{-\beta} D_t^{\alpha} u_n$ follows from the equality $t^{-\beta} D_t^{\alpha} u_n = a_i^2 \frac{\partial^2 u_n}{\partial x^2}$. Now we need to prove that the series (24) converges uniformly everywhere in $\bar{\Omega}$. Similarly, integrating by parts twice formula (25) we obtain $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{M_6}{\lambda_n}$. Thus, the majorizing series converges absolutely, which means that the series (24) converges uniformly in $\bar{\Omega}$ and defines a continuous function $u(x, t)$ in $\bar{\Omega}$. Thus, we have proven the existence of a solution to problem (1)-(4).

Now let's prove the uniqueness of the solution. Suppose there are two solutions $\nu(x, t)$, $w(x, t)$. Then for the function $\omega(x, t) = \nu(x, t) - w(x, t)$ we have the following problem:

$$t^{-\beta} D_t^{\alpha} \omega = a_i^2 \frac{\partial^2 \omega}{\partial x^2}, \text{ in the domain } \Omega = \cup \Omega_i, \ (i = 1,2), \tag{30}$$

$$\omega(x, 0) = 0, \quad 0 \leq x \leq l, \tag{31}$$

$$\begin{cases} a_{11}\omega_x(0, t) + a_{12}\omega(0, t) = 0, \\ a_{21}\omega_x(l, t) + a_{22}\omega(l, t) = 0, \end{cases} \quad 0 \leq t \leq T \tag{32}$$

$$\begin{cases} \omega(x_0 - 0, t) = \omega(x_0 + 0, t), \\ k_1 \frac{\partial\omega(x_0 - 0, t)}{\partial x} = k_2 \frac{\partial\omega(x_0 + 0, t)}{\partial x}. \end{cases} \tag{33}$$

The solution to this problem (30)-(33) can be represented as an expansion in the basis $\{X_n(x)\}$ and has the form

$$\omega(x, t) = \sum_{n=1}^{\infty} B_n(t) X_n(x). \tag{34}$$

The coefficients $B_n(t)$ are easy to find if we multiply both parts of the equality (34) by the functions $Y_n(x)$ respectively, and integrate the resulting ratio from 0 to l and take into account the biorthogonality of the sequences $\{X_n(x)\}$ and $\{Y_n(x)\}$. Then we get

$$B_n(t) = \int_0^l \omega(x, t) Y_n(x) dx. \tag{35}$$

We find the fractional derivative with t respect to equality (35), and taking into account equation (30) we have:

$$t^{-\beta} D_t^{\alpha}\left(B_n(t)\right) = \int_0^l t^{-\beta} D_t^{\alpha} \omega Y_n(x) dx =$$
$$= C_n a_1^2 \int_0^{x_0} \frac{k_1}{a_1^2} \omega_{xx} \Phi_2(x_0, \lambda_n) \Phi_1(x, \lambda_n) dx +$$
$$+ C_n a_2^2 \int_{x_0}^l \frac{k_2}{a_2^2} \omega_{xx} \Phi_1(x_0, \lambda_n) \Phi_2(x, \lambda_n) dx.$$

Integrating by parts twice and using the boundary conditions (32), the conjugation conditions (33), we have

$$t^{-\beta} D_t^{\alpha}\left(B_n(t)\right) = -C_n \frac{\lambda_n k_1}{a_1^2} \int_0^{x_0} \omega(x, t) \Phi_2(x_0, \lambda_n) \Phi_1(x, \lambda_n) dx -$$
$$- C_n \frac{\lambda_n k_2}{a_2^2} \int_{x_0}^l \omega(x, t) \Phi_1(x_0, \lambda_n) \Phi_2(x, \lambda_n) dx = -\lambda_n B_n(t).$$

Hence

$$B_n(t) = b_n E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_n t^{\alpha+\beta}\right).$$

Substituting the found value $B_n(t)$ into formula (35), we obtain

$$\int_0^1 \omega(x,t)Y_n(x)dx = b_n E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_n t^{\alpha+\beta}\right). \quad (36)$$

Passing to the limit in equality (36) at $t \to 0$, which is possible due to continuity $\omega(x,t)$ in $\overline{\Omega}$, we have $\lim_{t\to 0}\int_0^1 \omega(x,t)Y_n(x)dx = 0 = B_n(0)$, therefore $b_n = 0$, $(n = 1,2,\dots)$.

Then from formula (34) we obtain $\omega(x,t) = 0$, it follows that $v(x,t) = w(x,t)$. The uniqueness of the solution is proven, thus the theorem is proved.

Let's consider the following example:

$$\mathcal{L}u \equiv \left\{\begin{array}{l} t^{-\beta}D_t^{\alpha}u(x,t) - 4u_{xx}(x,t), \ \ 0 < x < 1 \\ t^{-\beta}D_t^{\alpha}u(x,t) - u_{xx}(x,t), \ \ 1 < x < 2 \end{array}\right\} = 0, \quad (37)$$

in the domain

$$\Omega = \cup\,\Omega_i,\ \Omega_1 = \{(x,t): 0 < x < 1,\ \ 0 < t < T\},$$
$$\Omega_2 = \{(x,t): 1 < x < 2,\ \ 0 < t < T\},\ (i = 1,2),$$

with the initial condition

$$u(x,0) = \varphi(x) = \begin{cases} 2x - 1, & 0 \le x \le 1, \\ x, & 1 \le x \le 2, \end{cases} \quad (38)$$

and boundary conditions

$$\begin{cases} u_x(0,t) + 2u(0,t) = 0, \\ u_x(2,t) - \frac{1}{2}u(2,t) = 0, \end{cases} \ \ 0 \le t \le T, \quad (39)$$

and with conjugation conditions

$$\begin{cases} u(1-0,t) = u(1+0,t), \\ u_x(1-0,t) = 2u_x(1+0,t), \end{cases} \ \ 0 \le t \le T. \quad (40)$$

It is easy to see that the function

$$\varphi(x) = \begin{cases} 2x - 1, & 0 \le x \le 1, \\ x, & 1 \le x \le 2, \end{cases}$$

satisfies conditions (23) of our theorem. Then the solution to problem (37)-(40) has the form:

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n X_n(x) E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_n t^{\alpha+\beta}\right),$$

where

$$\varphi_n = \int_0^2 \varphi(x)Y_n(x)dx = 2C_n\Phi_1(1,\lambda_n)\Phi_2'(1,\lambda_n) - \frac{1}{4}C_n\Phi_2(1,\lambda_n)\Phi_1'(1,\lambda_n),$$

$$X_n(x) = C_n\begin{cases} \Phi_2(1,\lambda_n)\Phi_1(x,\lambda_n), & 0 < x < 1, \\ \Phi_1(1,\lambda_n)\Phi_2(x,\lambda_n), & 1 < x < 2, \end{cases}$$

$$Y_n(x) = C_n\begin{cases} \frac{1}{4}\Phi_2(1,\lambda_n)\Phi_1(x,\lambda_n), & 0 < x < 1, \\ 2\Phi_1(1,\lambda_n)\Phi_2(x,\lambda_n), & 1 < x < 2, \end{cases}$$

$$C_n = \left(\frac{1}{4}\Phi_2^2(1,\lambda_n)\int_0^1 \Phi_1^2(x,\lambda_n)dx + 2\Phi_1^2(1,\lambda_n)\int_1^2 \Phi_2^2(x,\lambda_n)dx\right)^{-\frac{1}{2}},$$

$$\Phi_1(x,\lambda_n) = \cos\left(\frac{\sqrt{\lambda_n}}{2}x\right) - \frac{4}{\sqrt{\lambda_n}}\sin\left(\frac{\sqrt{\lambda_n}}{2}x\right),$$

$$\Phi_2(x,\lambda_n) = \cos\left(\sqrt{\lambda_n}(2-x)\right) - \frac{1}{2\sqrt{\lambda_n}}\sin\left(\sqrt{\lambda_n}(2-x)\right),$$

λ_n are roots of equation

$$\lambda_n\left(2\sin\sqrt{\lambda_n}\cos\frac{\sqrt{\lambda_n}}{2} + \frac{1}{2}\cos\sqrt{\lambda_n}\sin\frac{\sqrt{\lambda_n}}{2}\right) -$$

$$-\sqrt{\lambda_n}\left(\frac{25}{4}\sin\sqrt{\lambda_n}\sin\frac{\sqrt{\lambda_n}}{2}+3\cos\sqrt{\lambda_n}\cos\frac{\sqrt{\lambda_n}}{2}\right)-$$
$$-4\sin\sqrt{\lambda_n}\cos\frac{\sqrt{\lambda_n}}{2}-\cos\sqrt{\lambda_n}\sin\frac{\sqrt{\lambda_n}}{2}.$$

Conclusion

In this work, we considered an initial-boundary value problem for a two-phase heat conduction model on an interval $(0, l)$ with a strictly internal interface point $x = x_0$, Sturm-type boundary conditions, and a Caputo fractional derivative of order $0 < \alpha < 1$ in time, multiplied by a weight $t^{-\beta}$. The mismatch of thermophysical parameters at the point was taken into account through ideal contact (conjugation) conditions expressing the continuity of temperature and heat flux.

The method of separation of variables reduces the problem to a spectral boundary value problem for a second-order differential operator with a discontinuous coefficient at the highest derivative. We derived the corresponding characteristic equation determining the eigenvalues and constructed explicit forms of the eigenfunctions for both the original and the adjoint spectral problems. Since the original spectral problem is not self-adjoint, the orthogonality of the eigenfunctions is absent. To overcome this difficulty, we constructed an auxiliary self-adjoint problem with the same eigenvalues and proved that its eigenfunctions form an orthonormal basis in the space $L_2(0, l)$. Establishing a bounded and boundedly invertible transformation between the eigenfunctions of the self-adjoint and original problems, we proved that the eigenfunctions of the original spectral problem form a Riesz basis in $L_2(0, l)$.

Using this basis property and the explicit representation of time-dependent multipliers via the generalized Mittag-Leffler function, we obtained the solution of the initial-boundary value problem in the form of a Fourier-type series. We also justifiably differentiated the series term-by-term and proved the existence and uniqueness of the solution under natural smoothness and consistency conditions for the initial data. The results provide a rigorous justification of the Fourier method for a diffusion model with a time-fractional operator in a composite (two-phase) medium with discontinuous coefficients.

Possible directions for further research include extending the analysis to multilayered media with several interfaces, to more general conjugation or boundary conditions (including nonlocal ones), as well as to direct problems with sources and associated inverse problems on determining unknown coefficients or interface parameters.

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[1,2]Койлышов У.,
ф.-м.ғ.к., қауымдастырылған профессор, ORCID ID: 0000-0003-1752-7848,
e-mail: koylyshov@math.kz
[1,2]*Жаппарова С.,
докторант, ORCID ID: 0009-0007-4580-0983,
*e-mail: zhapparova.saule@gmail.com
[3]Ниетбаев А.,
ф.-м.ғ.к., қауымдастырылған профессор, ORCID ID: 0009-0006-5677-5464,
e-mail: a.niet57@mail.ru

[1]Математика және математикалық модельдеу институты, Алматы қ., Қазақстан
[2]Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы қ., Қазақстан
[3]М.Х. Дулати атындағы Тараз өңірлік университеті, Тараз қ., Қазақстан

УАҚЫТ БОЙЫНША БӨЛШЕК ТУЫНДЫСЫ БАР ЖӘНЕ ШТУРМ ТИПТІ ШЕКАРАЛЫҚ ШАРТТАРМЕН БЕРІЛГЕН ЖЫЛУӨТКІЗГІШТІКТІҢ ЕКІ ФАЗАЛЫ ЕСЕПТЕРІ

Аңдатпа

Бұл жұмыста уақыт бойынша $0 < \alpha < 1$ ретті Капуто туындысы бар, бөлікті-тұрақты коэффициенті бар жылуөткізгіштік теңдеуі үшін Штурм типті бастапқы-шекаралық есеп

зерттеледі. Қарастырылып отырған облыс $x = x_0$ қатаң ішкі үзіліс нүктесін қамтитын интервал $(0, l)$. Физикалық тұрғыдан алғанда, бұл есеп ұзындығы жіңішке өзектегі температура өрісінің таралу процесін модельдейді. Өзек құрама және әртүрлі жылуфизикалық сипаттамалары бар екі бөліктен – $(0, x_0)$ және (x_0, l) тұрады. Екі ортаның жанасу $x = x_0$ нүктесінде идеал жанасу шарттары қойылады, яғни бір ортадан екінші ортаға өткенде температураның үзіліссіздігін және жылулық ағынның үзіліссіздігін білдіреді. Жұмыстың негізгі мақсаты – айнымалыларды ажырату әдісімен (Фурье әдісімен) қойылған бастапқы-шекаралық есептің шешімін негіздеу. Бұл әдісті қолдану коэффициенті үзілісті жоғары ретті жәй дифференциалдық операторға сәйкес спектрлік есепті зерттеу қажеттілігіне әкеледі.

Түйін сөздер: жылуөткізгіштік теңдеу, бөлшек туындылар, үзілісті коэффициенттер, меншікті мәндер, меншікті функциялар, айнымалыларды ажырату әдісі.

[1,2]**Койлышов У.,**
к.ф.-м.н., ассоциированный профессор, ORCID ID: 0000-0003-1752-7848,
e-mail: koylyshov@math.kz
[1,2*]**Жаппарова С.,**
докторант, ORCID ID: 0009-0007-4580-0983,
*e-mail: zhapparova.saule@gmail.com
[3]**Ниетбаев А.,**
к.ф.-м.н., ассоциированный профессор, ORCID ID: 0009-0006-5677-5464,
e-mail: a.niet57@mail.ru

[1]Институт математики и математического моделирования, г. Алматы, Казахстан
[2]Казахский национальный университет им. аль-Фараби, г. Алматы, Казахстан
[3]Таразский региональный университет им. М.Х. Дулати, г. Тараз, Казахстан

ДВУХФАЗНЫЕ ЗАДАЧИ ТЕПЛОПРОВОДНОСТИ С ГРАНИЧНЫМИ УСЛОВИЯМИ ТИПА ШТУРМА И С ДРОБНОЙ ПРОИЗВОДНОЙ ПО ВРЕМЕНИ

Аннотация

В работе исследуется начально-краевая задача типа Штурма для уравнения теплопроводности с кусочно-постоянным коэффициентом и производной Капуто порядка $0 < \alpha < 1$ по времени. Рассматриваемая область представляет собой интервал $(0, l)$, содержащий строго внутреннюю точку разрыва $x = x_0$. С физической точки зрения эта задача моделирует процесс распространения температурного поля в тонком стержне длины l. Стержень является составным и состоит из двух участков – $(0, x_0)$ и (x_0, l), обладающих различными теплофизическими характеристиками. В точке контакта двух сред $x = x_0$ задаются условия идеального контакта, подразумевающие непрерывность температуры и непрерывность теплового потока при переходе из одной среды в другую. Основной целью работы является обоснование решения поставленной начально-краевой задачи методом разделения переменных (методом Фурье). Применение этого метода приводит к необходимости исследования соответствующей спектральной задачи для обыкновенного дифференциального оператора с разрывным коэффициентом при старшей производной.

Ключевые слова: уравнение теплопроводности, разрывные коэффициенты, собственные функции, собственные значения, дробные производные, метод разделения переменных.