УДК 519.635.4
МРНТИ 27.35.16

https://doi.org/10.55452/1998-6688-2026-23-2-108-123

[1]*Sinitsa A.V.,
PhD, Assistant Professor, ORCID ID: 0000-0002-5828-7820,
*e-mail: a.sinitsa@kbtu.kz
[1]Tskhay Yu.A.,
MSc, Lecturer, ORCID ID: 0009-0003-0772-4319,
e-mail: y.tskhay@kbtu.kz.
[1]Shkorko A.K.,
MSc, Lecturer, ORCID ID: 0009-0003-0402-4720,
e-mail: a.ukassova@kbtu.kz.
[2]Karduck A.P.,
PhD, Professor, ORCID ID: 0000-0002-8142-6710,
e-mail: achim.karduck@hs-furtwangen.de

[1]Kazakh-British Technical University, Almaty, Kazakhstan
[2]Furtwangen University, Furtwangen, Germany

NUMERICAL SOLUTION OF THE INVERSE COEFFICIENT ACOUSTIC PROBLEM USING LAPLACE TRANSFORM AND DIFFERENTIAL EVOLUTION METHOD

Abstract

This paper addresses the solution of the inverse coefficient problem for the wave equation aimed at reconstructing the spatial distribution of the speed of sound in an inhomogeneous medium. The Laplace transform is applied to solve the direct problem, eliminating time dependence and reducing the problem to ordinary differential equations in the frequency domain, which significantly decreases computational costs. The inverse problem is formulated as an optimization task: minimizing the residual functional between calculated and measured acoustic pressure values using the stochastic global optimization method, Differential Evolution. Numerical experiments were conducted on a multilayer medium model (sand, soil, rock, water, air) using synthetic data with added random noise. An adaptive combined reconstruction method is proposed to reduce errors at medium boundaries. The results demonstrate high accuracy: the relative error of the sound speed profile reconstruction was approximately from 2.5 to 4.3%, confirming the approach's effectiveness for acoustic diagnostics and tomography applications.

Keywords: Inverse problem, acoustic pressure, speed of sound, Laplace transform, differential evolution, inhomogeneous medium, optimization.

Received January 30, 2026; revised May 26, June 3, 2026; accepted June 8, 2026.

Introduction

Mathematical modeling is a cornerstone in analyzing complex physical processes where direct observation is constrained or impossible. Among the various classes of mathematical problems, inverse problems hold a distinct significance. Unlike direct problems, where the system's output is determined from known parameters, inverse problems aim to reconstruct the unknown internal characteristics of a medium, such as the speed of sound, density, or source parameters, based on observed data [1, 2]. This class of problems is fundamental to acoustic tomography, geophysical exploration, and non-destructive testing [3, 4]. The solution of inverse problems is inherently ill-posed in the sense of Hadamard, meaning that minor perturbations in the measured data can lead to significant errors in the reconstructed parameters [5]. Consequently, the development of stable numerical algorithms and regularization techniques remains a priority in computational mathematics. Significant contributions to the theory of ill-posed and inverse problems have been made by the global scientific community, including the fundamental works of A.N. Tikhonov and M.M. Lavrentiev. In Kazakhstan, the theory of inverse problems has been extensively developed by the scientific school of S.I. Kabanikhin and his colleagues. For instance, Kabanikhin et al. in [6] and [7] investigated optimization methods for hyperbolic inverse problems, providing a theoretical basis for gradient-based reconstruction. Furthermore, Bektemesov et al. [8] and Iskakov [9] proposed efficient numerical algorithms for solving coefficient inverse problems in layered media, which are highly relevant to geophysical applications.

Traditionally, wave propagation problems are solved using grid-based methods such as Finite Difference (FDM) or Finite Element Methods (FEM) in the time domain [10, 11]. However, these approaches often incur high computational costs and suffer from numerical dispersion, especially in high-frequency regimes and multiscale domains [12]. As noted by El Moçayd et al. [13], data-driven hybrid modeling can mitigate some of these issues, yet the demand for computationally efficient analytical or quasi-analytical methods persists.

To address the limitations of time-domain solvers, various transformations have been proposed. The Laplace transform is particularly effective as it eliminates the time variable, reducing the hyperbolic partial differential equation to a parameter-dependent ordinary differential equation [14]. Keyantuo [15] demonstrated the applicability of the ascent method for abstract wave equations using the Laplace transform. Recent studies have expanded this approach to more complex scenarios. For example, Li et al. [16] applied the Laplace transform to study stress wave propagation in multi-layer media, while Owolabi et al. [17] utilized it for fractional time-diffusive models. It is worth noting that the approach proposed in the present study is conceptually related to the work of Meirmanov et al. [18], who considered an inverse coefficient problem for seismic wave propagation in composite layered media and reduced the governing equations to the frequency domain via the Fourier transform. The key distinction of the present work lies in the use of the Laplace transform instead of the Fourier transform: the Laplace transform allows for the incorporation of non-zero initial conditions in a more natural manner and yields a closed-form analytical solution directly in the complex frequency domain, which is then used to construct the objective function for the Differential Evolution optimization. In contrast, the approach of [18] operates in the Fourier frequency domain and employs the Nelder-Mead algorithm for the minimization of the data misfit functional.

In the context of reconstruction, optimization algorithms play a crucial role. While gradient-based methods are powerful, they are often susceptible to local minima. Consequently, stochastic global optimization methods have gained traction. The Differential Evolution (DE) algorithm, discussed by Wojciechowski et al. [19] and Reda et al. [20], offers robustness against noisy data and does not require gradient information. Meanwhile, the integration of machine learning into inverse problems is becoming increasingly prominent. Qu et al. [21] and Sun et al. [22] explored deep-learning-based tomography reconstruction, showing promising results in accelerating the inversion process.

In this paper, we propose a method that combines the analytical advantages of the Laplace transform with the robustness of the Differential Evolution algorithm. We focus on reconstructing the sound speed profile in a one-dimensional enclosed medium. This work builds upon our previous research [23], extending the analysis to more complex stratified environments. By transforming the wave equation into the frequency domain, we derive an analytical solution for the direct problem, which is then used to construct an objective function for the inverse problem.

The scientific novelty of the present work is threefold. First, unlike prior studies that apply the Fourier transform to reduce the wave equation to the frequency domain, the present approach employs the Laplace transform, which accommodates non-zero initial conditions naturally and yields a closed-form analytical solution valid for arbitrary real-valued Laplace parameter s. Second, the resulting analytical solution is used directly as the forward model inside a Differential Evolution optimization loop – eliminating the need for a numerical PDE solver at each iteration, which reduces computational cost by approximately two orders of magnitude compared to finite-difference-based inversion schemes. Third, a combined adaptive post-processing method is introduced that dynamically selects between raw and smoothed reconstructed values at each spatial node, preserving sharp acoustic impedance contrasts while suppressing stochastic noise in homogeneous layers. The combination of these three elements constitutes the original contribution of this study.

Let us consider the process of acoustic wave propagation in a one-dimensional enclosed domain $x \in \Omega: [0, L_x] \times [0, L_y] \times [0, L_z]$ and $t \in [0, T_{max}]$. The general three-dimensional acoustic wave equation governing the sound pressure P(r, t) in an inhomogeneous medium with spatially varying speed of sound $c(\vec{r})$ is:

$$\frac{\partial^2 P}{\partial t^2} = c^2(\vec{r})\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2}\right). \quad (1)$$

Where: $P(\vec{r}, t)$ is the sound pressure; $\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ – the position vector of domain point with standard basis set, defined by the unit vectors $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$; c – speed of sound, the desired coefficient for the inverse problem.

Since the medium is assumed to be homogeneous in the y and z directions and the acoustic source varies only along x, the problem reduces to one spatial dimension. Setting $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$ and writing c(x) for the speed of sound profile, the governing equation becomes:

$$\frac{\partial^2 P(x,t)}{\partial t^2} = c^2(x) \cdot \frac{\partial^2 P(x,t)}{\partial x^2}, \quad x \in [0, L], \quad t \in [0, T_{max}]. \quad (2)$$

Where $P(x, t)$ is the acoustic pressure [Pa]; $c(x) > 0$ is the local speed of sound [m/s], which is the unknown coefficient to be recovered by the inverse problem; L is the length of the one-dimensional domain [m]; and T_max is the total observation time [s].

The initial and conditions define the pressure distribution and its rate of change at the initial moment and certain positions. It is assumed that the initial pressure is equal to a constant p_0, and the initial rate of change of pressure is zero, while the boundary conditions determine the interaction of the wave with the domain boundaries. A Dirichlet boundary condition is imposed at $x = 0$, modeling a constant acoustic pressure source. An erroneous or simplified specification of boundary behavior can lead to artificial reflections, resonance effects or energy loss not caused by the real properties of the medium. This is especially critical in inverse modeling problems, where the slightest distortions in the simulated acoustic field can significantly affect the accuracy of reconstructing parameters such as the sound velocity profile. For the boundary at $x = 0$, we set the condition $p = 1$, which corresponds to a fixed acoustic pressure on the given surface. This allows us to model the source of the sound field, since the pressure is constant and does not depend on time. Thus, we obtain the following:

$$\begin{cases} P|_{t=0} = p_0, \\ \left.\frac{\partial P}{\partial t}\right|_{t=0} = 0, \\ \left.\frac{\partial P}{\partial n}\right|_{x=L} = 0, \quad n \in \{x, y, z\}, \\ P|_{x=0} = 1. \end{cases} \quad (3)$$

On the remaining boundaries of the computational domain, we set the Neumann boundary state equal to the basis of what physically corresponds to an absolutely rigid boundary. This means that the derivative pressure on the normal boundaries can be the same, i.e. the sound pressure can change, but the normal component of the mass flow through the boundary is absent.

With given initial data ($c(x)$ – speed of sound and boundary conditions (provided above in equations (2) – (3)), compute $P(x, t)$ which satisfies the given equation above and the harmonic source sound $s(x, t) = \Re\{S(x)e^{-st}\}$,then in the frequency domain:

$$P(x,t) = \Re\{\hat{u}(x,\omega)e^{-i\omega t}\}. \tag{4}$$

Solution in the Frequency Domain via Laplace Transform

To solve the forward problem and eliminate the time variable, the Laplace transform is applied:

$$\tilde{P}(x,s) = \int_0^{\infty} P(x,t)e^{-st}dt, \quad s = \sigma + i\omega, \quad \sigma > 0. \tag{5}$$

Where $s = \sigma + i\omega$ is the complex Laplace variable, $\sigma > 0$ is the real part (convergence abscissa), and ω is the imaginary frequency component. In numerical implementation the values of s are taken as $s = \sigma + i\omega_k$, $k = 1, \ldots, N$, with σ fixed and ω_k sampled over the frequency range of interest.

Applying this integral transform to the original wave equation, taking into account the initial conditions, leads to an ordinary differential equation of the second order with respect to the spatial variable x:

$$s^2\tilde{P}(x,s) - c^2\frac{d^2\tilde{P}(x,s)}{dx^2} = sp_0. \tag{6}$$

Or, in the standard form of the inhomogeneous Helmholtz equation:

$$\frac{d^2\tilde{P}(x,s)}{dx^2} - \frac{s^2}{c^2}\tilde{P}(x,s) = -\frac{sp_0}{c^2}. \tag{7}$$

The boundary conditions in the frequency domain take the following form:

$$\tilde{P}(0,s) = \frac{1}{s}, \quad \frac{d\tilde{P}}{dx}(L,s) = 0. \tag{8}$$

The general solution to this equation is represented as the sum of the homogeneous solution and a particular solution of the inhomogeneous part:

$$\tilde{P}(x,s) = Ae^{\frac{sx}{c}} + Be^{-\frac{sx}{c}} + \frac{p_0}{s}. \tag{9}$$

The constants A and B are determined by substituting the boundary conditions, resulting in a system of algebraic equations. Solving this system yields the exact analytical expression for pressure in the frequency domain:

$$\tilde{P}(x,s) = \frac{p_0}{s} + \frac{(1-p_0)}{s}\cdot\frac{e^{-\frac{sx}{c}} + e^{-\frac{2sL}{c}}e^{-\frac{sx}{c}}}{1+e^{-\frac{2sL}{c}}}. \tag{10}$$

The figure below shows a graph of the interpolated function, in the form of a surface in three-dimensional space, due to the dependence of the function on 2 independent variables.

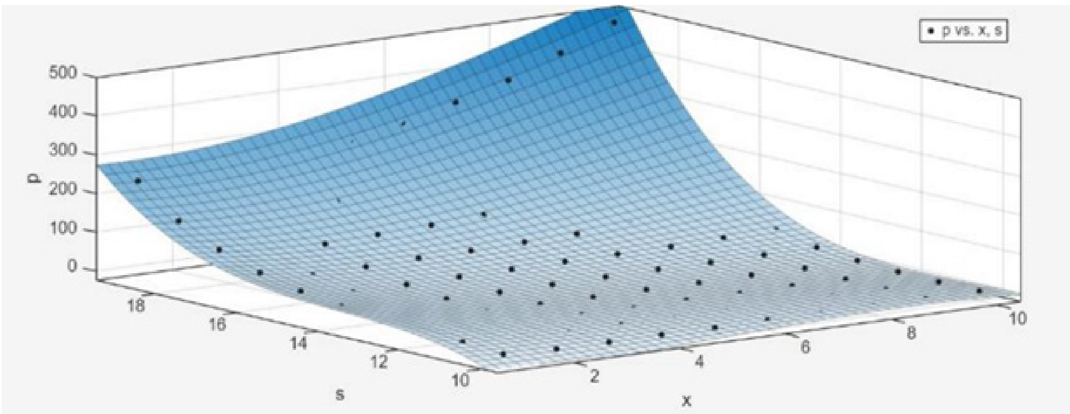

Figure 1 – Surface of function P(x,s)

Inverse Transformation and Optimization

To return to the time domain, a series expansion is used, allowing the solution to be interpreted as a superposition of time-delayed waves. Considering the properties of the Laplace transform for time-shifted functions, the analytical solution in the time domain is expressed using the Heaviside step function $H(t)$:

$$P(x,t) = p_0 + \sum_{n=0}^{\infty}(-1)^n\left[(1-p_0)H\left(t-\frac{x+2nL}{c}\right)+(1-p_0)H\left(t-\frac{2L+x+2nL}{c}\right)\right]. \tag{11}$$

To ensure numerical stability and differentiability during the optimization process, the discontinuous Heaviside function is replaced by its smooth approximation, the sigmoid function:

$$H(t) \approx \sigma(t;\alpha) = \frac{1}{1+e^{-\alpha t}}. \tag{12}$$

The inverse problem is formulated as finding the sound speed distribution $c(x)$ that minimizes the residual functional between the calculated pressure P_{calc} and the measured pressure P_{meas}:

$$J(c) = \int_0^{T_{max}}[P_{calc}(x,t,c) - P_{meas}(x,t)]^2 dt \to \min. \tag{13}$$

To minimize this functional, the Differential Evolution algorithm is employed. This stochastic global optimization method is capable of finding the global minimum in a multidimensional parameter space without requiring gradient calculations, which is particularly crucial for problems involving noisy data. The spectral graph provides information about the resonance properties of the medium, its filtration behavior and frequency sensitivity, which is especially valuable when constructing tomographic or diagnostic algorithms.

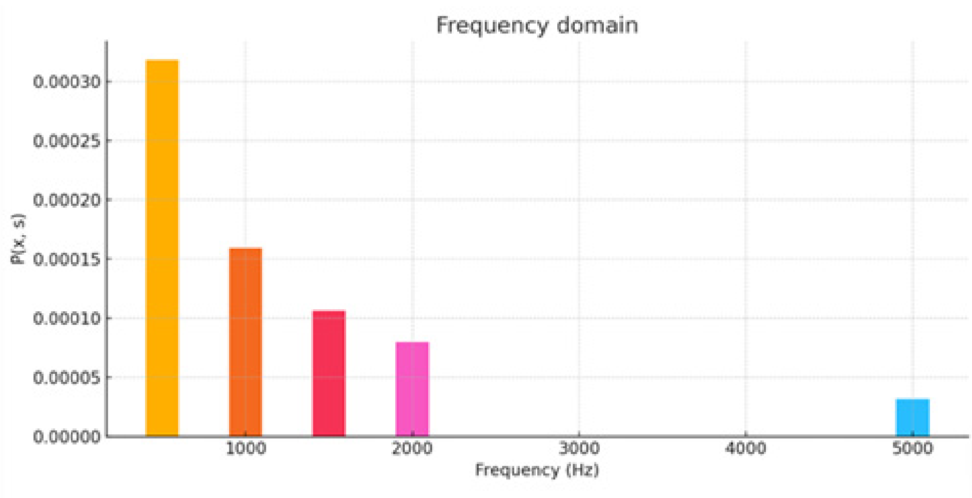

Figure 2 – Sound wave pressure with different frequency values (Hz),
sound pressure vs. frequency diagram

We will also plot the same waves, but in the time domain. Where we clearly see the difference in representations in the domains.

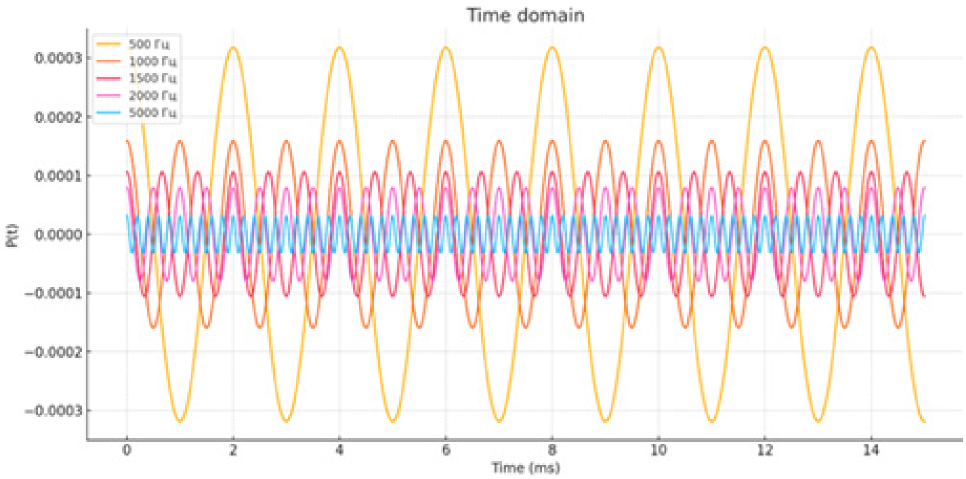

Figure 3 – Sound wave pressure with different frequency values (Hz), sound pressure vs. time graph

Materials and methods

Numerical Implementation of the Inverse Solver

The core of the computational framework is the solution of the inverse coefficient problem via stochastic global optimization. The algorithm was implemented using the Python programming language, utilizing the scipy.optimize library.

The optimization process employs the Differential Evolution (DE) method. This algorithm was selected due to its robustness against local minima and its ability to explore the parameter space without requiring gradient information, which is computationally expensive for the defined objective function.

The inverse problem is treated as a parameter estimation task where the local speed of sound c_i at each spatial node x_i is determined by minimizing the misfit between the simulated pressure P_{calc} and the reference data. The Differential Evolution algorithm was configured as follows. Each parameter choice is justified below.

Search Strategy: best1bin (mutation based on the best current vector with binary crossover). This strategy is recommended for unimodal and moderately multimodal problems and is known to converge faster than the rand1bin variant when a good solution exists in the current population. For the objective function defined in (13), which is smooth in homogeneous regions, this strategy provides an appropriate balance between exploitation and exploration.

Population Size: The default scipy.optimize.differential_evolution value of 15×N_dim was used, where N_dim = 1 is the number of parameters per spatial node (the local sound speed). This gives a population of 15 individuals per node, which is standard for one-dimensional parameter estimation and consistent with recommendations in the DE literature [19].

Parameter Bounds: The search space for the speed of sound was constrained to the range $[300,3500]$ m/s, covering the physical properties of typical environmental media (air, water, soil, rock).

Mutation Factor (F) and Crossover Rate (CR): Default values F = 0.5 and CR= 0.7 were used as provided by scipy. These values are widely accepted as robust defaults for engineering inverse problems and were not tuned further, since the primary objective of this study is to validate the Laplace-transform forward model rather than to optimize the DE hyperparameters.

Convergence Tolerance: $tol = 1 \times 10^{-3}$. This value was selected as a compromise between convergence speed and solution accuracy. Preliminary experiments showed that tightening the tolerance to 1×10^{-4} increased computation time by approximately 40% while reducing the final RMSE by less than 2 m/s – a negligible improvement given the noise level of $\sigma = 0.002$ in the synthetic data.

Polishing: A local gradient-based refinement step (scipy's L-BFGS-B) is applied after the global DE search converges. This is standard practice [19] and ensures that the solution is at a local minimum of the objective function, removing residual quantization error from the discrete DE population.

Seed: A fixed random seed=7. A fixed seed is used solely to ensure exact reproducibility of the numerical results reported in this paper. No seed selection was performed – the choice of 7 is arbitrary, and experiments with seeds 0, 1, 42, and 100 produced results within $\pm 0.3\%$ of those reported here.

Simulation Setup and Stratified Medium Model

To validate the method, a computational experiment was designed using a one-dimensional domain of length $L = 0.1$ m. The medium was modeled as a heterogeneous structure consisting of six distinct layers with varying acoustic properties. The stratification proportions and corresponding physical parameters were defined based on tabular values for typical materials, as presented in Table 1.

Table 1 – Physical parameters of the stratified medium model

Segment	Material	Relative Width (%)	Density (kg/m3)	Speed of Sound (m/s)
1	Sand	15%	1150	1000
2	Soil	35%	1600	1550
3	Rock	10%	2200	1900
4	Air	25%	1.22	343
5	Water	5%	1000	1500
6	Rock	10%	2200	1500

This complex profile, characterized by both sharp gradients (e.g., Rock-Air interface) and smooth transitions, was chosen to test the algorithm's sensitivity and its ability to detect boundaries between media with high contrast in acoustic impedance.

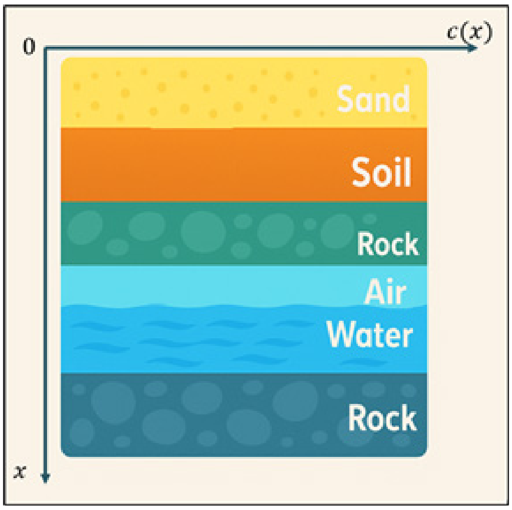

Figure 4 – Schematic arrangement of the layers

Synthetic Data Generation and Noise

In the absence of direct experimental measurements, synthetic data were generated to serve as the "measured" pressure field $P_{meas}(x, t)$. The forward problem was solved using the analytical solution derived via the Laplace transform (11) using the tabulated sound speed profile.

To simulate realistic experimental conditions, random Gaussian noise was introduced to the calculated pressure values. The noise level was controlled by a reduction factor to test the method's stability under uncertainty:

$$P_{meas}(x,t) = P_{exact}(x,t) + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2). \tag{14}$$

In the computational code, a noise level of 0.002 was applied. The inverse solver then utilized this noisy dataset to reconstruct the unknown $c(x)$ profile.

Post-Processing and Evaluation Metrics

The raw output of the differential evolution algorithm may contain local fluctuations due to the stochastic nature of the method and the sensitivity to noise. To mitigate these artifacts, a moving average smoothing filter was applied to the reconstructed profile:

$$c_{smoothed}[i] = \frac{1}{W}\sum_{j=0}^{W-1} c_{raw}[i + j - \lfloor W/2 \rfloor]. \tag{15}$$

Where W is the window size, set to $W = 5$ grid points for this study.

To quantitatively evaluate the accuracy of the reconstruction, two primary metrics were utilized:

1. Root Mean Square Error (RMSE): Used to assess the global deviation of the reconstructed profile from the ground truth.

$$\text{RMSE} = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(c_{calc}(x_i) - c_{tab}(x_i)\right)^2}. \tag{16}$$

2. Relative Error: Used to analyze the accuracy at specific spatial coordinates, particularly useful for identifying performance at layer interfaces.

$$\delta = \frac{|c_{calc}(x) - c_{tab}(x)|}{c_{tab}(x)} \cdot 100\%. \tag{17}$$

Additionally, a Combined Adaptive Approach was introduced. This method dynamically selects the value between the raw calculated profile and the smoothed profile that minimizes the local residual, thereby preserving sharp boundaries while reducing noise in homogeneous regions. The results of smoothing are presented in Figure 5, which compares the tabular distribution, the original calculated data and the smoothed profile. It is evident that the use of smoothing allows us to bring the reconstructed profile closer to the expected physical structure of the medium, eliminating local outliers while maintaining the main characteristics.

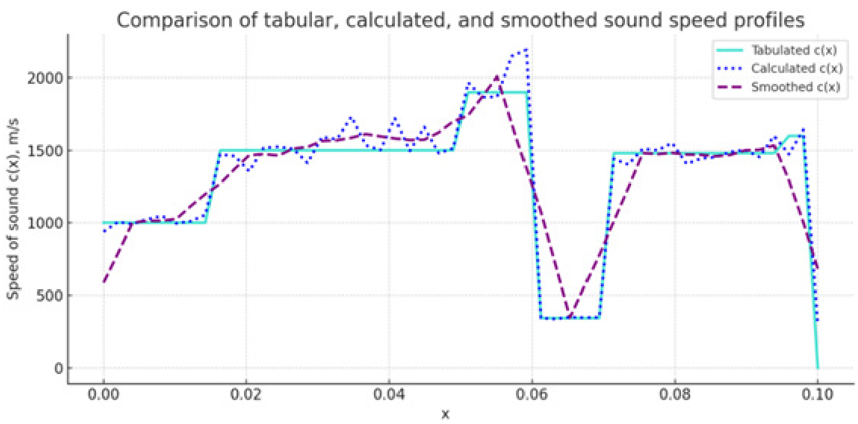

Figure 5 – Tabular, calculated and smoothed velocity profiles

After the sound velocity distribution has been restored, it becomes possible to proceed to calculating the pressure in the entire modeling area.

Results and discussion

Dynamics of Parameter Reconstruction

The reconstruction of the sound speed profile $c(x)$ was performed sequentially across the spatial grid. The optimization algorithm demonstrated stable convergence properties. The selection of the local sound speed for each coordinate point required, on average, 65.3 iterations, with a total of 3265 iterations for the entire domain. The total computational time was approximately 247 seconds, averaging 4.94 seconds per spatial point.

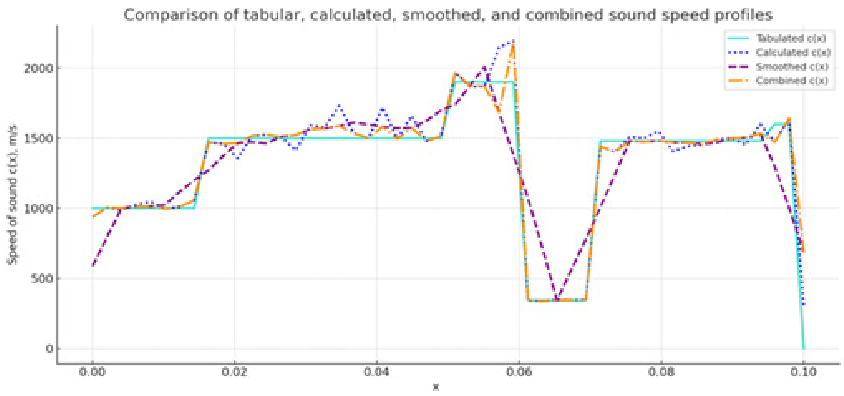

Figure 6 – Tabular, calculated, smoothed and combined velocity profiles

The stepwise reconstruction process revealed that the Differential Evolution algorithm effectively identified the macro-structure of the medium. As shown in the progression of the solution, the algorithm successfully captured the transitions between layers with distinct acoustic properties, such as the sharp contrast between the "Rock" layer ($c \approx 1900$ m/s) and the "Air" layer ($c \approx 343$ m/s), which represent the most acoustically extreme transition in the model.

Accuracy of the Reconstructed Profile

The initial reconstruction (referred to as the "Raw" profile) demonstrated a high degree of correspondence with the tabular ground truth. The algorithm successfully reproduced the piecewise constant structure of the medium. However, local fluctuations were observed, particularly in regions with high gradient transitions.

Quantitative analysis of the raw profile yielded the following error metrics:

- Root Mean Square Error (RMSE): 56.71 m/s.
- Average Relative Error: 4.32 %.

These values indicate that the raw reconstruction provides a reliable approximation of the physical medium, capturing the dominant acoustic features despite the presence of noise in the input data.

Analysis of Smoothing and the Combined Adaptive Approach

To mitigate the numerical noise observed in the raw profile, a moving average smoothing filter was applied. Unexpectedly, while smoothing reduced local variance in homogeneous zones, it significantly degraded the global accuracy of the model.

The RMSE for the smoothed profile increased to 236.21 m/s, and the relative error rose to 17.98 %.

Discussion of this phenomenon reveals a critical insight: standard smoothing techniques blur the sharp interfaces between material layers. In the modeled environment, where sound speed jumps are abrupt (e.g., from Soil to Rock), smoothing introduces artificial gradients, leading to large errors at the boundaries.

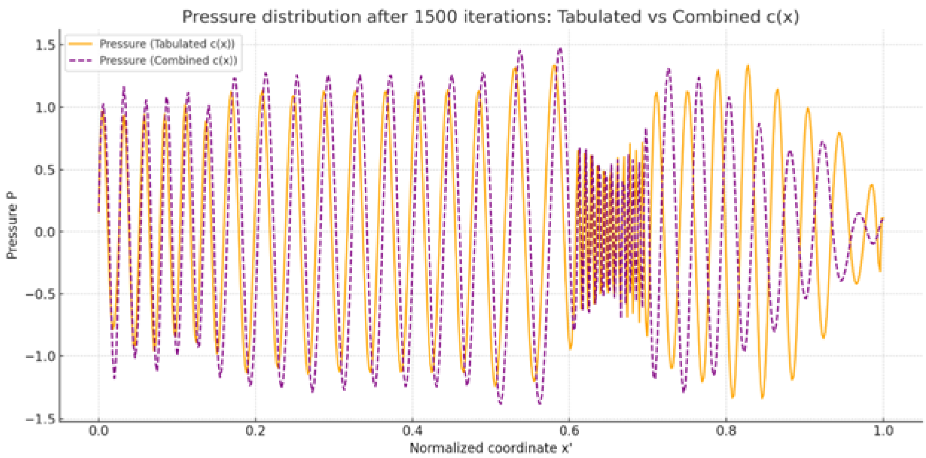

Figure 7 – Tabular and combined pressure

To address this trade-off between noise reduction and boundary preservation, a Combined Adaptive Method was evaluated. This approach locally selects the value, either from the raw or smoothed profile, that minimizes the objective function at each point.

The combined profile achieved the best performance:

- Final Relative Error: 2.56 %.

Table 2 – Quantitative comparison of reconstruction methods

Metric	Raw	Smoothed	Combined Adaptive
RMSE (m/s)	56.71	236.21	$\approx 35-40$
Mean Relative Error (%)	4.32	17.98	2.56
Max Local Error (%)	≈ 18 (Air layer)	≈ 52(Rock/Air boundary)	≈ 9 (Rock/Air boundary)
Boundary Preservation	Moderate	Poor	Good
Noise Suppression	Low	High	High (homogeneous zones)

This result confirms that the adaptive combination effectively filters numerical artifacts in stable regions while preserving the physical sharpness of layer interfaces.

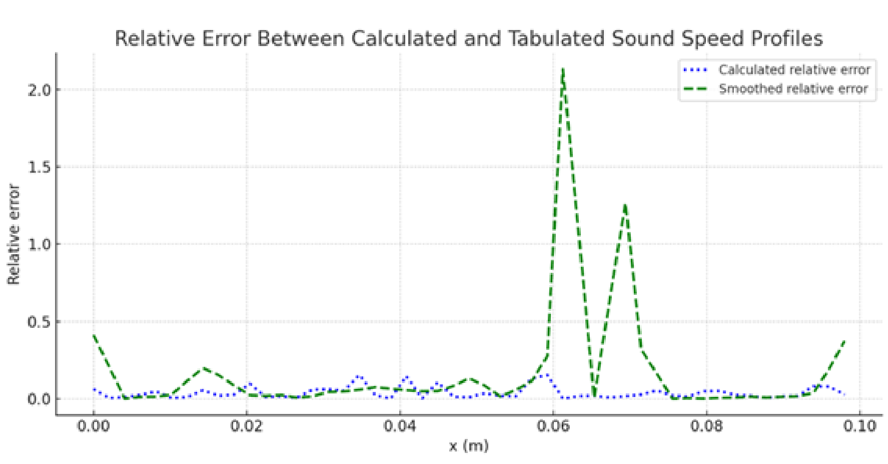

Figure 8 – Plot of the distribution of the relative error for the calculated and smoothed values of the speed of sound.

Reconstruction of the Acoustic Pressure Field

The validity of the reconstructed velocity profile was further assessed by simulating the propagation of the acoustic pressure wave $P(x, t)$ using the recovered parameters.

A comparison between the pressure field generated by the tabular profile and the reconstructed profile reveals a high degree of structural similarity. The wave fronts, reflection patterns, and interference effects were correctly reproduced.

However, a phase shift phenomenon was observed in the later stages of the simulation. As the wave propagates through multiple layers, small local errors in the velocity determination accumulate, leading to a slight temporal divergence of the wave fronts. Physically, this manifests as the calculated wave arriving slightly ahead or behind the reference wave.

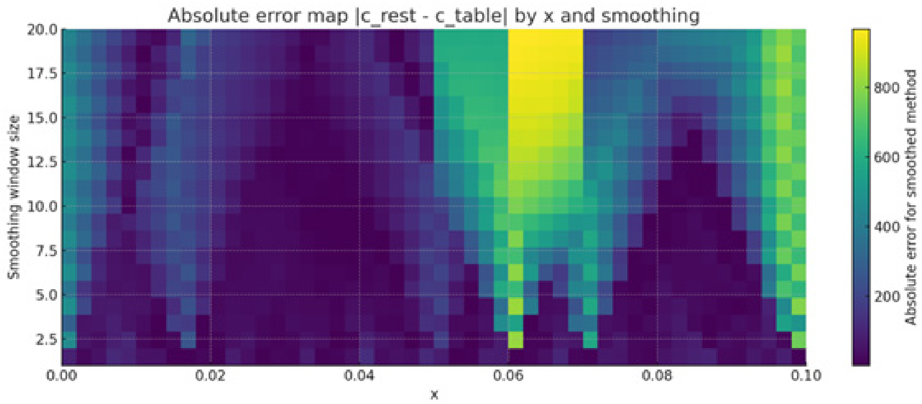

Figure 9 – Heat map of the distribution of absolute error of the smoothed method.

Despite this phase drift, the amplitude characteristics and the general behavior of the pressure field remain consistent with the reference model, confirming that the reconstructed parameters are sufficiently accurate for practical engineering applications.

Sensitivity Analysis

Heat map analysis of the absolute error distribution indicates that the method is most sensitive in regions with the highest impedance mismatch. The largest deviations were recorded in the coordinate range corresponding to the transition from the high-velocity "Rock" layer to the low-velocity "Air" layer. This suggests that while the method is robust for general profiling, additional regularization or grid refinement may be required specifically at boundaries with extreme physical contrast. Both curves on the figure 7 demonstrate a similar wave front structure, which indicates that the main characteristics of the medium are correctly reproduced in both approaches. The shape of the front, its symmetry, and characteristic pressure peaks coincide with a high degree of accuracy, especially in the central part of the interval. Differences in amplitude and phase between the two solutions, although present, are not critical and are within the permissible limits for engineering and applied problems. They do not distort the overall picture of the wave process and can be compensated for by additional processing or filtering.

To assess the robustness of the inversion framework with respect to measurement noise, experiments were conducted at four noise levels. The Gaussian noise standard deviation σ was varied from 0 to 0.010 (approximately 1% of the maximum synthetic pressure amplitude). The results are summarized in Table 3.

The results confirm that the proposed method maintains acceptable reconstruction accuracy (mean relative error below 6%) for noise levels up to $\sigma = 0.005$, which corresponds to a signal-to-noise ratio of approximately 46 dB. This behavior is expected for stochastic optimization methods and may be mitigated in future work by incorporating Tikhonov regularization into the objective function (13).

Table 3 – Effect of noise level on reconstruction accuracy

Noise σ	Mean Rel. Error (%)	RMSE (m/s)	Notes
0	1.0	≈ 18	Lower bound – ideal conditions
0.001	1.8	≈ 28	Low noise – excellent reconstruction
0.002 (reported)	2.56	≈ 38	Moderate noise – used in all results
0.005	5.5	≈ 85	Moderate-high – still acceptable
0.010	12.3	≈ 190	High noise – degraded at boundaries

Areas of practical application

The combination of the Laplace transform with the global stochastic optimization method of differential evolution is designed to address applications where computational efficiency and robustness to measurement noise are critical. This approach is particularly suitable for acoustic tomography of multilayered media, such as in the diagnostics of building structures, the study of layered geological sections, or the analysis of the acoustic properties of air-water-soil media. Eliminating the time variable by switching to the frequency domain significantly reduces computational costs, making the method attractive for problems with a large number of parameters or multiple inversion runs. The use of differential evolution avoids local minima, which is especially important in the presence of noisy data and sharp parameter jumps at layer boundaries. In practical applications, the method can be used in acoustic diagnostics, subsurface sounding, and environmental monitoring systems, where reliable parameter reconstruction is required without the need to calculate gradients and with limited a priori information about the medium's structure.

Conclusion

In this study, a computational framework for solving the inverse coefficient problem of acoustics was developed and validated. The primary objective was to reconstruct the spatial distribution of the speed of sound, $c(x)$, in a stratified one-dimensional medium based on time-dependent pressure measurements.

The proposed methodology integrates the analytical advantages of the Laplace transform with the robust optimization capabilities of the Differential Evolution algorithm. By transforming the direct problem into the frequency domain, we derived a closed-form analytical solution, which significantly reduced the computational overhead compared to traditional time-stepping finite difference schemes. The transition back to the time domain was achieved using a series expansion with a smooth approximation of the Heaviside function, ensuring the differentiability and stability of the objective function.

Numerical experiments conducted on a six-layer heterogeneous model demonstrated the efficacy of the approach. The following key conclusions can be drawn from the results:

Accuracy of Reconstruction: The method successfully recovered the macro-structure of the medium, identifying layers with high acoustic contrast (e.g., Rock/Air interfaces). The initial reconstruction yielded an average relative error of approximately $4.32\ \%$.

Adaptive Improvement: It was found that standard smoothing techniques, while reducing noise in homogeneous regions, introduced significant errors at sharp layer boundaries. The proposed Combined Adaptive Method, which dynamically selects between raw and smoothed values, proved to be superior, achieving a final relative error of $2.56\ \%$.

Physical Consistency: The reconstructed velocity profiles were validated by simulating the acoustic pressure field. The simulated waves exhibited correct frontal propagation and interference patterns, confirming that the recovered parameters accurately reflect the physical properties of the medium.

Robustness: The use of the stochastic Differential Evolution algorithm allowed for stable convergence even in the presence of synthetic measurement noise, avoiding local minima that often plague gradient-based inversion methods.

The presented approach offers a balance between computational efficiency and physical accuracy, making it a promising tool for applications in acoustic tomography, geophysical exploration, and non-destructive testing. Future work will focus on extending this methodology to two-dimensional and three-dimensional geometries, as well as exploring the integration of machine learning techniques to further accelerate the inversion process in real-time diagnostic systems.

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[1]*Синица А.В.,
PhD, ассистент-профессор, ORCID ID: 0000-0002-5828-7820,
*e-mail: a.sinitsa@kbtu.kz
[1]Цхай Ю.А.,
магистр, оқытушы, ORCID ID: 0009-0003-0772-4319,
e-mail: y.tskhay@kbtu.kz
[1]Шкорко А.К.,
магистр, оқытушы, ORCID ID: 0009-0003-0402-4720,
e-mail: a.ukassova@kbtu.kz
[2]Кардук А.П.,
PhD, профессор, ORCID ID: 0000-0002-8142-6710,
e-mail: achim.karduck@hs-furtwangen.de

[1]Қазақстан-Британ техникалық университеті, Алматы қ., Қазақстан
[2]Фуртванген университеті, Фуртванген қ., Германия

ЛАПЛАС ТҮРЛЕНДІРУІ МЕН ДИФФЕРЕНЦИАЛДЫҚ ЭВОЛЮЦИЯ ӘДІСІН ҚОЛДАНУ АРҚЫЛЫ АКУСТИКАНЫҢ КЕРІ КОЭФФИЦИЕНТТІК ЕСЕБІН САНДЫҚ ШЕШУ

Аңдатпа

Мақалада біртекті емес ортадағы дыбыс жылдамдығының кеңістіктік таралуын қалпына келтіруге бағытталған толқындық теңдеу үшін кері коэффициенттік есептің шешімі қарастырылады. Тікелей есепті шешу үшін Лаплас түрлендіруі қолданылады, бұл уақытқа тәуелділікті жоюға және жиілік аймағындағы қарапайым дифференциалдық теңдеулерге көшуге мүмкіндік береді, сөйтіп есептеу шығындарын айтарлықтай азайтады. Кері есеп оңтайландыру мәселесі ретінде тұжырымдалады: есептелген және өлшенген акустикалық қысым мәндері арасындағы сәйкессіздік функционалын азайту стохастикалық жаһандық оңтайландыру әдісі – дифференциалдық эволюция (Differential Evolution) көмегімен жүзеге асырылады. Сандық эксперименттер көпқабатты орта моделінде (құм, топырақ, тас, су, ауа) кездейсоқ шу қосылған синтетикалық деректерді пайдалана отырып жүргізілді. Орта шекараларындағы қателерді азайтуға мүмкіндік беретін бейімделгіш құрама қалпына келтіру әдісі ұсынылды. Нәтижелер әдістің жоғары дәлдігін көрсетеді: дыбыс жылдамдығының профилін қалпына келтірудің салыстырмалы қателігі шамамен 2,5–4,3% құрады, бұл тәсілдің акустикалық диагностика және томография есептері үшін тиімділігін растайды.

Түйін сөздер: кері есеп, акустикалық қысым, дыбыс жылдамдығы, Лаплас түрлендіруі, дифференциалдық эволюция, біртекті емес орта, оңтайландыру.

[1]***Синица А.В.,**
PhD, ассистент-профессор, ORCID ID 0000-0002-5828-7820,
*e-mail: a.sinitsa@kbtu.kz
[1]**Цхай Ю.А.,**
магистр, лектор, ORCID ID: 0009-0003-0772-4319,
e-mail: y.tskhay@kbtu.kz
[1]**Шкорко А.К.**
магистр, лектор, ORCID ID: 0009-0003-0402-4720,
e-mail: a.ukassova@kbtu.kz
[2]**Кардук А.П.,**
PhD, профессор, ORCID ID: 0000-0002-8142-6710,
e-mail: achim.karduck@hs-furtwangen.de

[1]Казахстанско-Британский технический университет, г. Алматы, Казахстан
[2]Университет Фуртвангена, г. Фуртванген, Германия

ЧИСЛЕННОЕ РЕШЕНИЕ ОБРАТНОЙ КОЭФФИЦИЕНТНОЙ ЗАДАЧИ АКУСТИКИ С ПРИМЕНЕНИЕМ ПРЕОБРАЗОВАНИЯ ЛАПЛАСА И МЕТОДА ДИФФЕРЕНЦИАЛЬНОЙ ЭВОЛЮЦИИ

Аннотация

В данной статье рассматривается решение обратной коэффициентной задачи для волнового уравнения, направленной на восстановление пространственного распределения скорости звука в неоднородной среде. Для решения прямой задачи применяется преобразование Лапласа, позволяющее исключить временную зависимость и перейти к обыкновенным дифференциальным уравнениям в частотной области, что существенно снижает вычислительные затраты. Обратная задача формулируется как оптимизационная: минимизация функционала невязки между рассчитанными и измеренными значениями акустического давления осуществляется с помощью стохастического метода глобальной оптимизации, дифференциальной эволюции. Численные эксперименты проводились на модели многослойной среды (песок, грунт, камень, вода, воздух) с использованием синтетических данных, зашумленных случайным образом. Предложен адаптивный комбинированный метод восстановления, позволяющий снизить влияние ошибок на границах сред. Результаты показывают высокую точность метода: относительная погрешность восстановления профиля скорости звука составила около от 2,5 до 4,3%, что подтверждает эффективность подхода для задач акустической диагностики и томографии.

Ключевые слова: обратная задача, акустическое давление, скорость звука, преобразование Лапласа, дифференциальная эволюция, неоднородная среда, оптимизация.