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ON UNIVERSAL NUMBERINGS OF TWO-ELEMENT FAMILIES IN THE ERSHOV HIERARCHY

Abstract

The study of local and global invariants of the Rogers semilattice is an important and fundamental problem in numbering theory and computability theory. Global invariants include properties such as an existence of a universal numbering, the number of minimal numberings, the cardinality of the entire semilattice, and a criterion for determining whether a semilattice is a lattice. Local invariants, in turn, describe structures, such as initial segments or intervals within the semilattice. We say that a numbering $\nu \in Com(S)$ is universal if any other numbering $\mu \in Com(S)$ reduces to ν. The study of universal numberings is important for understanding the structure of semilattices and their classification. In this paper, an existence of universal numberings is considered for finite families of computably enumerable sets located at finite levels of the Ershov hierarchy. The main result is that for any two-element family of computably enumerable sets S, its Rogers semilattice, considered at the third level of the Ershov hierarchy, has universal numberings.

Keywords: computable numberings, universal numberings, Rogers semilattice, Ershov hierarchy.

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Introduction

The study of local and global invariants of the Rogers semilattice is one of the central problems of numbering theory. Among the global invariants are an existence of a universal numbering, the number of minimal numberings, the cardinality of the semilattice, and its lattice property. Local invariants characterize the structure of initial segments and intervals, in particular, their finiteness or density.

One of the important global invariants is an existence of a universal numbering. Recall that a numbering is called universal if any other numbering in the same family can be reduced to it via a computable function. In other words, there exists an algorithm that effectively reproduces all other algorithms that enumerate a given family. For families of computably enumerable (c.e.) sets, an existence of a universal numbering was established in the work of Yu. L. Ershov [1].

Algorithmic enumerations of elements of Σ^{-1}_1-sets are monotonic: if an element is enumerated into a set at some step of the computation, it remains in this set thereafter. Therefore, the growth of the set being enumerated is monotonic. The same thing happens in enumerations of Σ^0_n-sets for any level n; the only difference from Σ^{-1}_1-enumerations is that suitable oracles are used. In algorithms for enumerating Σ^{-1}_n-sets for $n > 1$, monotonicity is absent: initially, no number is present in the set; at some step of the algorithm, a number may be enumerated into the set; this is the first change in the number's status; later, this number may be extracted from the set; this is the second change; then, possibly, the number is enumerated into the set again, this is the third change; and so on. Sets in the Ershov hierarchy are classified by the number of "mind changes" of the procedures of inclusion in the set or exclusion from the set of each of the elements of the set [2]. For all numbers in the set from the class Σ_n^{-1}, this number of mind changes is bounded by the number n. Thus, the class Σ_1^{-1} consists of computably enumerable sets, and the class Σ_2^{-1} consists of differences of computably enumerable sets. Even the first studies of computable numberings of families of sets from the class Σ_2^{-1} showed the difference between the fundamental invariants of their Rogers semilattices compared to the Rogers semilattices of families of Σ_1^{-1} sets.

By S. Badaev and Zh. Talasbayeva, a family was constructed consisting of one computably enumerable set and one Σ_2^{-1}-set, related by an inclusion relation, which had a one-element Rogers semilattice [3]. This is impossible in the classical case: the Rogers semilattice of any finite family of Σ_1^{-1}-sets containing at least one pair of nested sets is infinite and contains both the greatest and the least elements. At the same time, there exists a finite family of Σ_2^{-1}-sets whose Rogers semilattice does not have a greatest element constructed by K. Abeshev [4].

The main result of the paper is the following statement. For any two-element family of computably enumerable sets belonging to an odd level of the Ershov hierarchy, there exists a universal numbering.

The paper consists of an introduction, preliminaries and one chapter, which provides a complete proof of an existence of a universal numbering for the two-element family of computably enumerable sets on the third level of the Ershov hierarchy.

Materials and methods

In this paper we will follow to the notations and terminology adopted in [1], [5]. We define the pair $\langle \cdot,\cdot \rangle$ as a computable bijection from ω^2 to ω that is called Cantor's pairing function

$$\langle x, y\rangle = \frac{(x+y)^2 + 3x + y}{2}$$

For any $x, y \in \omega$ we can define the functions l and r, such that $x = \langle l(x), r(x)\rangle$, $l(\langle x, y\rangle) = x$ and $r(\langle x, y\rangle) = y$.

Let n be a non-zero natural number. The class Σ_n^{-1} of the Ershov hierarchy contains all n-c.e. sets. A set $B \subseteq \omega$ is n-c.e. (or n-computably enumerable) if $B = \lim_s A_s$ for a uniformly computable sequence of sets A_s such that $A_0 = \emptyset$, and for each $x \in \omega$, there are at most n-many s satisfying $A_s(x) \neq A_{s+1}(x)$. Here we say that the function $f(x, s) = A_s(x)$ is an n-approximation of the n -c.e. set B.

A numbering ν of some family $\mathcal{F} \in \Sigma_n^{-1}$ is called a Σ_n^{-1}-computable numbering if the set

$$\{\langle x, k\rangle: \ x \in \nu(k), \ k \in \omega\}$$

is a n-c.e. set. We say that a numbering ν is reducible to a numbering μ (denoted as $\nu \leq \mu$) if there is a computable function f such that $\nu = \mu \circ f$. We say that ν and μ are equivalent ($\nu \equiv \mu$) if $\nu \leq \mu$ an $\mu \leq \nu$. $\mathrm{Com}_n^{-1}(\mathcal{S})$ denotes the set of Σ_n^{-1}-computable numberings of the family $\mathcal{S}$. For numberings ν and μ their join ($\nu \oplus \mu$) is defined as

$$(\nu \oplus \mu)(2k) = \nu(k) \text{ and } (\nu \oplus \mu)(2k+1) = \mu(k).$$

The quotient structure $\mathcal{R}_n^{-1}(\mathcal{S}) = \{\mathrm{Com}_n^{-1} \backslash_{\equiv}, \leq, \oplus\}$ is called a Rogers semilattice of the Σ_n^{-1} -computable family $\mathcal{S}$.

A numbering ν is called Σ_n^{-1}-universal numbering if $\nu \in \mathrm{Com}_n^{-1}(\mathcal{S})$ and any numbering $\mu \in \mathrm{Com}_n^{-1}(\mathcal{S})$ is reduced to ν.

Results and discussion

In this section we will consider two-element family of c.e. sets $\mathcal{S}$.

Theorem 1. For any two-element family of c.e. sets $\mathcal{S} = \{A, B\}$, the Rogers semilattice $\mathcal{R}_3^{-1}(\mathcal{S})$ has a universal numbering.

Proof. We divide the proof into two cases:

Case 1: one of the sets is a subset of the other. Without loss of generality, we can assume that $A \subsetneq B$. In this case, we fix some element $b \in B \setminus A$ and computable approximations $(A_s)_{s\in\omega}$ and $(B_s)_{s\in\omega}$ for c.e. sets A and B, respectively. Moreover, in this case, we can assume that $A_s \subseteq B_s$ for all $s \in \omega$. Then we define the numbering $\nu(x) := \lim_s V_{x,s}$ where

$$V_{x,s} = \begin{cases} A_s, & \text{if } b \notin \pi_s(x), \\ B_s, & \text{if } b \in \pi_s(x). \end{cases}$$

by $(\pi_s(x))_{s\in\omega}$ we define aprroximation of a universal numbering of all 3-c.e. sets.

Lemma 2. The constructed numbering $\nu \in \mathrm{Com}_3^{-1}(\mathcal{S})$.

Proof. For any $x \in \omega$, the sequence $V_{x,s}$ coincides with either A_s or B_s at each step. Therefore, three options are possible:

$\lim_s V_{x,s} = A$;

$\lim_s V_{x,s} = B$;

the limit does not exist.

The third option is impossible, since it would imply an infinite number of changes in the status of the element b in the set $\pi(x)$, which contradicts the 3-computability of $\pi(x)$.

This means we have only two options: - $\nu(x) = A$ if $b \notin \pi(x)$ and - $\nu(x) = B$ if $b \in \pi(x)$. Therefore, ν is the numbering of the family $S = \{A, B\}$.

Now we prove that for a fixed x the limit $\lim_s V_{x,s}$ is a 3-c.e. set. Let z be an arbitrary element. Consider the cases.

1. $z \notin A$ and $z \notin B$. In this case z will not be enumerated into $V_{x,s}$. No mind-changes.
2. $z \in B \setminus A$.

a. $\lim_s V_{x,s} = A$. The element b either never appears in $\pi(x)$ or appears at some step s_0 and then will be permanently removed on step s_1. If b never appears, then $V_{x,s} = A_s$ at all steps, and z never appears. If b appears at step s_0 and is permanently removed at step s_1, then z may appear before s_1 (if it is already listed in B) and will be permanently removed at step s_1. In this case, there are at most two changes.

b. $\lim_s V_{x,s} = B$. There exist steps $s_0 < s_1 < s_2$ such that b first enters π at step s_0, is then removed at step s_1, and finally enters at step s_2. If z already belongs to B before s_1, changes are possible at steps s_1 and s_2. If z enters B after s_1, it is added once and is not removed again. In all cases, there are no more than three changes.

$z \in A$ and $z \in B$.

a. $\lim_s V_{x,s} = A$. If b never occurs in $\pi(x)$, then $V_{x,s} = A_s$, and z appears once. If b occurs at step s_0 and is removed at step s_1, then:

- if z enters A before s_0, it can be temporarily removed and then added again;
- if z first enters B, it can be removed at s_1 and then added again later. In any case, there are at most three changes.

b. $\lim_s V_{x,s} = B$. Again, there are steps $s_0 < s_1 < s_2$. If z enters A before the corresponding switches, then, since $A_s \subseteq B_s$, it is no longer removed. If z first enters B, changes are possible at steps s_0, s_1, s_2. The number of changes does not exceed three.

Since $V_{x,s}$ is constructed uniformly in x, s, the numbering ν is 3-computable.

Lemma 3. The numbering ν is a universal numbering in $\mathcal{R}_3^{-1}(\mathcal{S})$.

Proof. Since π is universal, for every computable numbering $\mu \in \mathrm{Com}_3^{-1}(\mathcal{S})$ there exists a computable function g such that

$$\mu(x) = \pi(g(x)).$$

By construction, $\nu(g(x)) = \pi(g(x))$ and $\nu(g(x)) = \mu(x)$ for all x. Hence $\mu \leq \nu$, and therefore ν is universal in $\mathcal{R}_3^{-1}(\mathcal{S})$. ☐

Case 2. Suppose that A is not a subset of B, and vice versa. Fix elements $a \in A \backslash B$ and $b \in B \backslash A$.

Let π_e be a fixed universal computable numbering of all 3-c.e. families. For all e we will build the numbering ν_e and function f_e.

Construction.

Stage 0. For all x we set $\nu_{e,0}(x) = \emptyset$ and $f_{e,0}(x) \uparrow$.

Stage $s+1$. Let $l(s+1) = y$ at each step. Check is $f_{e,s}(y) \downarrow$?

1. If $f_{e,s}(y)$ is already defined, let $f_{e,s}(y) = i$. Then we check what stage the module for (e, y) is at, and continue its execution.

2. If $f_{e,s}(y)$ is undefined, then we check whether stage s marks the first occurrence of one of the markers a or b in $\pi_e(y)$, that is:

- either $a \in \pi_{e,s}(y)$ and previously $a \notin \pi_{e,t}(y)$ and $b \notin \pi_{e,t}(y)$ for all $t < s$,
- or $b \in \pi_{e,s}(y)$ and previously $b \notin \pi_{e,t}(y)$ and $a \notin \pi_{e,t}(y)$ for all $t < s$.

If one of them occurs for the first time, then we fix a new index i, define $f_{e,s+1}(y) = i$ and run the module for the pair (e, y) with this index i.

3. If none of these events occurred, then we set $\nu_{e,s+1}(y) = A_{s+1}$ and keep the function $f_{e,s+1}(y) \uparrow$.

Description of the module for (e, y) with index i.

1. Wait for a stage s_0 such that either $a \in \pi_{e,s_0}(y)$ or $b \in \pi_{e,s_0}(y)$. Until such a stage appears, keep $\nu_{e,s}(i) = A_s$.

2. Suppose $a \in \pi_{e,s_0}(y)$. Set $\nu_{e,s_0}(i) = A_{s_0}$. Wait for a stage $s_1 > s_0$ such that $a \notin \pi_{e,s_1}(y)$ and $b \in \pi_{e,s_1}(y)$.

3. At stage s_1, redefine

$$\nu_{e,s_1}(i) = B_{s_1},$$

that is, remove all elements of $A_{s_1} \backslash B_{s_1}$. Then wait for a stage $s_2 > s_1$ such that $b \notin \pi_{e,s_2}(y)$ and $a \in \pi_{e,s_2}(y)$.

4. At stage s_2, redefine

$$\nu_{e,s_2}(i) = A_{s_2},$$

that is, remove all elements of $B_{s_2} \backslash A_{s_2}$, and for all steps $t > s_2$ $\nu_{e,t}(i) = A_t$.

If at stage s_0 we instead have $b \in \pi_{e,s_0}(y)$, the construction is defined symmetrically, interchanging the roles of a and b, and of A and B.

We now analyze the possible outcomes.

1. If the construction waits forever at step (1), then neither a nor b ever enumerates in $\pi_e(y)$. In this case $\nu_e(i) = A$.

2. If the construction waits forever at step (2), then $a \in \pi_e(y)$ and $b \notin \pi_e(y)$. Hence $\pi_e(y) = A$, and $\nu_e(i) = A$.

3. If the construction waits forever at step (3), then $b \in \pi_e(y)$ and $a \notin \pi_e(y)$. Hence $\pi_e(y) = B$, and $\nu_e(i) = B$.

4. If the construction reaches step (4), then the last redefinition yields $\nu_e(i) = A$. In this case $\pi_e(y) = B$, but according to the strategy the numbering $\pi_e(y)$ will switch back to A.

The numbering ν we will define as follows: $\nu(\langle e, x \rangle) = \nu_e(x)$. ☐

Lemma 4. The constructed numbering $\nu \in \mathrm{Com}_3^{-1}(S)$.

Proof. For any $x \in \omega$, the sequence $\nu_{e,s}(x)$ coincides with either A_s or B_s at each step. Therefore, three options are possible:

1. $\lim_s \nu_{e,s}(x) = A$;
2. $\lim_s \nu_{e,s}(x) = B$;
3. the limit does not exist.

The third option is impossible, since it would imply an infinite number of changes in the status of the elements a and b in the set $\pi(x)$, which contradicts the 3-computability of $\pi(x)$.

This means we have only two options: - $\nu(x) = A$ if $a \in \pi(x)$ and - $\nu(x) = B$ if $b \in \pi(x)$, it is impossible to have both $a \in \pi(x)$ and $b \in \pi(x)$ because these elements distinguishes the sets A and B. Therefore, ν is a numbering of the family $S = \{A, B\}$.

Now we prove that for a fixed x the limit $\lim_s \nu_s(x)$ is a 3-c.e. set. Let z be an arbitrary element. Consider the cases.

1. $z \notin A$ and $z \notin B$. In this case z will not be enumerated into $\nu_s(x)$. No mind-changes.
2. $z \in B \backslash A$.

a. $\lim_s \nu_{e,s}(x) = A$. The element b either never appears in $\pi(x)$ or appears at some step s_0 and then will be permanently removed on step s_1 and the element a can be enumerated on step s_3, removed on s_4 and enumerated forever on some s_5. If b never appears, then $\nu_s(x) = A_s$ at all steps, and z never appears. If b appears at step s_0 and is permanently removed at step s_1, then z may appear before s_1 (if it is already listed in B), and will be permanently removed at step s_1. Or it can be enumerated after step s_1, then it will not appear in ν. In this case, there are at most two changes.

b. $\lim_s \nu_{e,s}(x) = B$. There exist steps $s_0 < s_1 < s_2$ such that b first enters π at step s_0, is then removed at step s_1, and finally enters at step s_2. And there exist steps $s_3 < s_4 < s_5$ such that a first enters π at step s_3, is then removed at step s_4, and finally enters at step s_5. If z already belongs to B before s_1, changes are possible at steps s_1 and s_2. If z enters B after s_1, it is added once and is not removed again. In all cases, there are no more than three changes.

3. $z \in A \backslash B$. Same as the previous one.
4. $z \in A$ and $z \in B$.

a. $\lim_s \nu_{e,s}(x) = A$. If b never occurs in $\pi(x)$, then $\nu_{e,s}(x) = A_s$, and z appears at step s_3 can be removed on s_4 and will be enumerated forever at s_5, so at most three mind-changes. If b occurs at step s_0 and is removed at step s_1, then:

- if z enters A before s_0, it can be temporarily removed and then added again;
- if z first enters B, it can be removed at s_1 and then added again later, since it's in intersection of sets, it will never be removed when we see $z \in A$ and $z \in B$. In any case, there are at most three changes.

b. $\lim_s \nu_{e,s}(x) = B$. If a never occurs in $\pi(x)$, then $\nu_{e,s}(x) = B_s$, and z appears at step s_0 can be removed on s_1 and will be enumerated forever at s_2, so at most three mind-changes. If a occurs at step s_3 and is removed at step s_4, then:

- if z enters B before s_3, it can be temporarily removed and then added again;
- if z first enters A, it can be removed at s_4 and then added again later, since it's in intersection of sets, it will never be removed when we see $z \in A$ and $z \in B$.. In any case, there are at most three changes.

Lemma 5. If the numbering $\pi_e \in \mathrm{Com}_3^{-1}(\mathcal{S})$, then the function f_e is total.

Proof. Let π_e enumerates the family $\mathcal{S}$. By construction, for each x, there exists a step s at which either the set A or the set B corresponding to the index x appears in π_e. At this point, the module (e, x) is activated, which determines the value of $f_e(x)$.

Therefore, for any x, the value of $f_e(x)$ is determined at some finite step, meaning that the function f_e is defined everywhere. Thus, f_e is a total function. □

Lemma 6. If the numbering $\pi_e \in \mathrm{Com}_3^{-1}(\mathcal{S})$, then $\forall x\, \pi_e(x) = \nu_e(f_e(x))$.

Proof. Consider the module (e, x) (see module analysis). By construction, if $\pi_e(x)$ enumerates the set A, then the module sets the value of $f_e(x)$ in such a way that $\nu_e(f_e(x))$ also enumerates A. Similarly, if $\pi_e(x)$ enumerates the set B, then $\nu_e(f_e(x))$ enumerates B.

Thus, in any case, we obtain the equality $\pi_e(x) = \nu_e(f_e(x))$. □

Lemma 7. The numbering ν is a universal numbering in $\mathcal{R}_3^{-1}(\mathcal{S})$.

Proof. Let $\pi_e \in \mathrm{Com}_3^{-1}(\mathcal{S})$. Then by Lemma 6 there exists a total function f_e such that for all x $\pi_e(x) = \nu_e(f_e(x))$.

Consequently, $\pi_e(x) = \nu(\langle e, f_e(x)\rangle)$.

Since f_e is a total function, $\nu(\langle e, f_e\rangle$ defines a computable reducibility of π_e to ν. Therefore, ν is a universal numbering in $\mathcal{R}_3^{-1}(\mathcal{S})$. ☐

Using the same module construction and the same reduction scheme, the result generalizes to all $2n + 1$ levels of the Ershov hierarchy: at each odd level, a universal numbering of the corresponding class is constructed by means of a similar parametrization and uniform reduction.

Corollary 8. For any family $\mathcal{S} = \{A, B\}$, where A, B are c.e. sets, $\mathcal{R}_{2n+1}^{-1}(\mathcal{S})$ has a universal numbering for every n.

Conclusion

This paper shows an existence of a universal numberings for the Rogers semilattice of two-element families of c.e. sets on odd levels of the Ershov hierarchy.

REFERENCES

1 Ershov, Y.L. Teoriya numeratsii [The Theory of Numberings] (Moscow: Nauka, 1977). (In Russian).

2 Ash, C.J., and Knight, J.F. Computable Structures and the Hyperarithmetical Hierarchy (Elsevier Science, 2000).

3 Badaev, S.A., and Talasbaeva, Zh.T. Computable numberings in the hierarchy of Ershov. In: Mathematical Logic in Asia, edited by S.S. Goncharov et al. (Singapore: World Scientific, 2006), pp. 17–30.

4 Abeshev, K.Sh. On the existence of universal numberings for finite families of d.c.e. sets. Mathematical Logic Quarterly, 60 (3), 161–167 (2014).

5 Soare, R.I. Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets (Berlin: Springer Science & Business Media, 1999).

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ЕРШОВ ИЕРАРХИЯСЫНДАҒЫ ЕКІ ЭЛЕМЕНТТІ ЖИЫНДАР ҮЙІРЛЕРІ ҮШІН УНИВЕРСАЛ НӨМІРЛЕУЛЕР ТУРАЛЫ

Аңдатпа

Роджерс жарты торының локалды және глобалды инварианттарын зерттеу – нөмірлеу теориясындағы маңызды әрі іргелі мәселелердің бірі. Глобалды инварианттарға универсал нөмірлеудің болуы, минималды нөмірлеулер саны, бүкіл жарты тордың қуаттылығы және жарты тордың тор болу критерийі сияқты қасиеттер жатады. Ал локалды инварианттар жарты тор ішіндегі бастапқы сегменттер мен аралықтар тәрізді құрылымдарды сипаттайды. Егер кез келген $\mu \in Com(S)$ нөмірлеуі $\nu \in Com(S)$ нөмірлеуіне көшірілсе, онда ν нөмірлеуі универсал балады. Универсал нөмірлеуді зерттеу жарты торлардың құрылымын және олардың жіктелуін түсіну үшін маңызды. Мақалада Ершов иерархиясының шекті деңгейлерінде орналасқан есептелімді жиындардың шекті үйірлері үшін универсал нөмірлеулердің болуы қарастырылады. Негізгі нәтиже – есептелімді жиындардың кез келген екі элементті S үйірі үшін Ершов иерархиясының үшінші деңгейінде қарастырылатын Роджерс жарты торында универсал нөмірлеудің бар екендігі дәлелденеді.

Түйін сөздер: есептелімді нөмірлеулер, универсал нөмірлеулер, Роджерс жарты торы, Ершов иерархиясы.

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ОБ УНИВЕРСАЛЬНЫХ НУМЕРАЦИЯХ ДЛЯ ДВУХЭЛЕМЕНТНЫХ СЕМЕЙСТВ В ИЕРАРХИИ ЕРШОВА

Аннотация

Изучение локальных и глобальных инвариантов полурешетки Роджерса является важной и фундаментальной задачей в теории нумерации и теории вычислимости. Глобальные инварианты включают такие свойства, как существование универсальной нумерации, число минимальных нумераций, мощность всей полурешетки и критерий определения того, является ли полурешетка решеткой. Локальные инварианты, в свою очередь, описывают структуры, такие как начальные сегменты или интервалы внутри полурешетки. Мы говорим, что нумерация $\nu \in Com(S)$ является универсальной, если любая другая нумерация $\mu \in Com(S)$ сводится к ν. Изучение универсальных нумераций важно для понимания структуры полурешеток и их классификации. В данной работе рассматривается существование универсальных нумераций для конечных семейств вычислимо перечислимых множеств, расположенных на конечных уровнях иерархии Ершова. Основной результат заключается в том, что для любого семейства вычислимо перечислимых множеств S, состоящего из двух элементов, его полурешетка Роджерса, рассматриваемая на третьем уровне иерархии Ершова, имеет универсальную нумерацию.

Ключевые слова: вычислимые нумерации, универсальные нумерации, полурешетка Роджерса, иерархия Ершова.