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## INVESTIGATION OF A MULTIPERIODIC PROBLEM FOR A FINITE-HEREDITARY SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS

### Abstract

Numerous problems arising in various fields of natural sciences and engineering lead to the study of integro-differential equations. The investigation of such equations, which account for the historical behavior of processes or phenomena, originates from the pioneering works of V. Volterra, where the role of the integral term was emphasized. Various approaches have since been developed to describe hereditary effects and aftereffects in these equations. It should be noted that the existence of multiperiodic and quasiperiodic solutions for systems of Volterra-type integro-differential equations does not always guarantee their uniqueness. In this paper, a system of partial integro-differential equations with a special differentiation operator is considered. Using the method of periodic characteristics, integral representations of the solution manifold for such systems with aftereffects are constructed. The properties of iterated kernels and resolvents are studied, and corresponding estimates are obtained. Conditions for the existence of a Green-type matrix function for the multiperiodic problem are established, along with integral representations with suitable bounds. Finally, sufficient conditions for the existence and uniqueness of a multiperiodic solution, interpreted as an integral manifold of the integro-differential system with finite hereditary effects, are derived.

**Keywords:** matricant, kernel, resolving operator, dichotomy, Green type function.

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### Introduction

Investigation of some processes in natural sciences and technology often necessitates the study of integro-differential equations.

Many problems in natural sciences and various applied fields are effectively modeled through mathematical frameworks described by integro-differential equations.

It is well established that certain problems of mathematical biology, biomechanics, control theory, population dynamics, elasticity theory, etc. lead to the study of partial integro-differential equations [1–8].

Of particular interest are processes where the dynamics depend not only on their current state but also on their previous states. The influence of the integral term on certain classes of differential equations and its impact on the nature of their solutions has been explored in [9–12]. The study of such processes with aftereffects is intricately linked to finite-period heredity. Notably, these processes and phenomena induce temporal oscillations [13–14].

Research efforts focusing on multiperiodic and almost periodic solutions of specific classes of partial integro-differential equations provide a detailed review and analysis of these problems, considering the role of finite hereditary effects [15–18].

In this context, developing methods to extend the oscillatory solutions of differential equation systems to integro-differential systems with specialized differentiation operators in multidimensional time and along vector field directions is of significant interest.

To study the problem of integrating multiperiodic functions along the periodic characteristics of the differentiation operator with respect to  $(\tau, t)$ , an approach is proposed [19–21]. By employing the linear transformation  $t = t_0 + \beta^*(\tau - \tau_0) \equiv \beta(\tau, \tau_0, t_0)$ , the  $\theta$ -periodic  $\beta$ -helical characteristic of the operator  $D_e$  is constructed, which describe translational and rotational motion, and the  $\theta$ -periodic by  $\tau$  characteristic  $\eta = \beta(\xi, \tau, \xi)$  is determined.

By linear substitution  $t_j = c_j \tilde{t}_j$ ,  $j = \overline{1, m}$  or  $t = c\tilde{t}$ , the differentiation operator  $D_e$  is reduced to operator  $D_c$  with  $m$ -dimensional vector  $e = (1, \dots, 1)$ , where  $c = \text{diag}[c_1, \dots, c_m]$  is a matrix,  $\tilde{t}_j = (\tilde{t}_1, \dots, \tilde{t}_m)$ .

Therefore, operator  $D_c$  has a  $\theta$ -periodic characteristic

$$\eta = c\beta(\xi, \tau, c^{-1}t) \equiv h(\xi, \tau, t).$$

The vector-function  $h(\xi, \tau, t)$  has the properties:

$$i) h(\xi + \theta, \tau, t) = h(\xi, \tau + \theta, t) = h(\xi, \tau, t),$$

$$ii) h(\xi, \tau, t + \omega) = h(\xi, \tau, t) + \omega,$$

it is periodic by  $t$  of the second kind with a period  $\omega = (\omega_1, \dots, \omega_m)$ .

For a specific class of partial integro-differential equations with a specialized differentiation operator, integral representations of the solution manifold are derived. The representation of solutions in terms of the kernel resolvent is analyzed, highlighting the critical role of these equations in modeling phenomena and processes characterized by aftereffects.

In contrast to the existing results on almost periodic and multiperiodic solutions of integro-differential equations, the present paper considers a system with a special differentiation operator  $D_c$  acting along periodic characteristics in multidimensional time [15–21].

The novelty of the approach consists in combining the method of periodic characteristics with the construction of a resolving operator based on the matricant and the kernel resolvent for systems with finite heredity. This makes it possible to obtain integral representations of solutions and to establish sufficient conditions for the existence and uniqueness of multiperiodic solutions in a more general setting.

## Materials and methods

I. Consider a vector integro-differential equation of the form

$$D_c u(\tau, t) = A(\tau, t)u(\tau, t) + \int_{\tau}^{\tau+\theta} K(\xi, \tau, t, h(\xi, \tau, t))u(\xi, h(\xi, \tau, t))d\xi, \quad (1)$$

where  $D_c = \frac{\partial}{\partial \tau} + \left\langle c, \frac{\partial}{\partial t} \right\rangle$  is differentiation operator;  $h(\xi, \tau, t)$  is periodic characteristic of the operator  $D_c$ ;  $A(\tau, t) \in C_{\tau, t}^{(0, \theta)}(R \times R^m)$  and  $K(\xi, \tau, t, h) \in C_{\xi, \tau, t, h}^{(0, 0, \theta, \theta)}(R \times R \times R^m \times R^m)$  are  $n \times n$ -matrices;  $e$  and  $\tilde{e}$  are vectors with single components. We assume that the conditions (P) are met if:

$$A(\tau + \theta, t + \omega) = A(\tau, t); \quad (2)$$

$$K(\xi + \theta, \tau, t, h(\xi + \theta, \tau, t)) = K(\xi, \tau, t, h(\xi, \tau, t));$$

$$K(\xi, \tau + \theta, t + \omega, h(\xi, \tau + \theta, t + \omega)) = K(\xi, \tau, t, h(\xi, \tau, t)), \quad (3)$$

where  $\omega_0 = \theta$ ,  $\omega = (\omega_1, \dots, \omega_m)$  is rationally incommensurable positive constants.

The conditions (P) express the multiperiodicity, smoothness, and boundedness of the matrix-function  $A(\tau, t)$  and the kernel  $K(\xi, \tau, t, h)$ . In particular, relations (2)-(3) ensure the invariance of the system with respect to shifts by the periods  $\theta$  and  $\omega$ , which is necessary for the construction of multiperiodic solutions along the periodic characteristics of the operator  $D_c$ .

Moreover, these conditions guarantee the boundedness of the corresponding operators and allow the application of the method of periodic characteristics and the construction of the resolving operator.

Similarly, condition (P<sub>1</sub>) means that the nonhomogeneous term  $f(\tau, t)$  possesses the same multiperiodicity properties and belongs to the same smoothness class as the coefficients of the system. This assumption is essential for preserving the multiperiodic structure of the solution.

Building upon the approach outlined in for constructing the matrix associated with the equation  $D_c v = A(\tau, t)v$ , we define the matrix  $V(\tau_0, \tau, t)$  which satisfies the matrix equation and reduces to the identity  $n \times n$ -matrix [17]. Utilizing the existence of the inverse operator  $D_c^{-1}$ , we research the matricant as a solution to the matrix equation expressed as a series, with its terms determined through recurrence relations. The absolute and uniform convergence of this series can be rigorously verified. The main properties of the matricant are discussed [17], [21]. Notably, one of the key attributes of the matricant  $V(\tau_0, \tau, t)$  is its multiperiodicity with respect to  $\tau_0$ ,  $\tau$  and  $t$ :

$$V(\tau_0 + \theta, \tau + \theta, t + \omega) - V(\tau_0, \tau, t) = 0.$$

Using the conditions of multiperiodicity (2) of the matrix  $A(\tau, t)$  and its smoothness with unknown vector-functions and the kernel  $K(\xi, \tau, t, h(\xi, \tau, t))$  of the hereditary term, the integro-differential system (1) by linear transformation

$$u(\tau, t) = V(\tau_0, \tau, t)w(\tau, t), \tag{4}$$

is reduced to a system of equations of the form

$$D_c w(\tau_0, \tau, t) = \int_{\tau}^{\tau+\theta} Q(\xi, \tau_0, \tau, t, h(\xi, \tau, t))w(\tau_0, h(\xi, \tau, t))d\xi. \tag{5}$$

Here  $Q(\xi, \tau_0, \tau, t, h(\xi, \tau, t)) = V^{-1}(\tau_0, \tau, t)K(\xi, \tau, t, h(\xi, \tau, t))V(\tau_0, \xi, h(\xi, \tau, t))$  is a kernel. Note that  $D_c V^{-1}(\tau_0, \tau, t) = -V^{-1}(\tau_0, \tau, t)A(\tau, t)$ , then  $V^{-1}(\tau_0, \tau, t)$  also has the properties of smoothness, multiperiodicity and boundedness. Due to the properties of matrices  $V(\tau_0, \tau, t)$  and  $V^{-1}(\tau_0, \tau, t)$ , as well as condition (3), the kernel has the property

$$\begin{aligned} Q(\xi + \theta, \tau_0 + \theta, \tau + \theta, t + \omega, h(\xi + \theta, \tau + \theta, t + \omega)) &= \\ &= Q(\xi, \tau_0, \tau, t, h(\xi, \tau, t)). \end{aligned}$$

The matricant  $W(\tau_0, \tau, t)$  of the equation (5) is defined as the solution of the matrix integral equation

$$\begin{aligned} W(\tau_0, \tau, t) &= E + \\ &+ \int_{\tau_0}^{\tau} \int_{\eta}^{\eta+\theta} Q(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t))W(\xi, s, h(\xi, \tau, t))d\xi d\eta \end{aligned} \tag{6}$$

Using the iterative method, its resolvent  $R(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t))$  is constructed, and satisfying the integral equation

$$\begin{aligned} R(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t)) &= Q(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t)) + \\ &+ \int_s^{\tau} \int_{\eta}^{\eta+\theta} Q(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t))R(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t))d\xi d\eta \end{aligned} \tag{7}$$

Thus, the resolvent (7) is well-defined and satisfies the corresponding integral equation in the class of bounded matrix-functions. It is worth noting that, based on the product of matrices and through calculations involving changes in the order of integration while considering the group property,

iterated kernels  $Q_m(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t))$  are constructed. These kernels are determined through recurrence relations, and their corresponding estimates are established

$$\|Q_m(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t))\| \leq Q_0^m \frac{\theta^{m-1}(\tau-\xi)^{m-1}}{(m-1)!}. \tag{8}$$

The integral equation (7) is obtained by successive approximations. Substituting the expression for  $R$  into the right-hand side and iterating, one obtains a series representation in terms of iterated kernels. The convergence of this series follows from estimate (8) and the boundedness of the kernel  $Q$ .

Taking into account (8), for the resolving kernel holds

$$\|R(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t))\| \leq Q_0 e^{Q_0 \theta(\tau-s)},$$

where  $\|Q(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t))\| \leq Q_0$ ,  $Q_0 = \text{const}$ ;  $s$  and  $\eta$  changes between  $\tau_0$  and  $\tau$ ,  $|\tau - \tau_0| < \Delta$ ,  $\Delta - \text{const} > 0$ .

Note that the resolvent  $R(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t))$  has the property of multiperiodicity:

$$\begin{aligned} R(\xi, s + \theta, \eta, h(\eta, \tau + \theta, t + \omega), h(\xi, \tau + \theta, t + \omega)) &= \\ &= R(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t)). \end{aligned}$$

Next, the matrix (6) is represented by the resolvent

$$\begin{aligned} \mathbb{W}(\tau_0, \tau, t) &= E + \\ &+ \int_{\tau_0}^{\tau} \int_{\eta}^{\eta+\theta} R(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t)) d\xi d\eta. \end{aligned} \tag{9}$$

Thus, taking into account the transformation (4) and the resolvents, we have

$$\Phi(\tau_0, \tau, t) = V(\tau_0, \tau, t) \mathbb{W}(\tau_0, \tau, t). \tag{10}$$

A matrix of the form (10) is called the resolving operator of the integro-differential equation (1). This operator  $\Phi(\tau_0, \tau, t)$  has the following properties:

- $i_1) D_c \Phi(\tau_0, \tau, t) = A(\tau, t) \Phi(\tau_0, \tau, t) + \int_{\tau}^{\tau+\theta} K(\xi, \tau, t, h(\xi, \tau, t)) \Phi(\xi, \tau_0, h(\xi, \tau, t)) d\xi$ ;
- $i_2) \Phi(\tau_0, \tau + \theta, t + \omega) = \Phi(\tau_0, \tau, t)$ ;
- $i_3) \|\Phi(\tau_0, \tau, t)\| \leq d e^{-\alpha(\tau-\tau_0)}$ ,  $\tau \geq \tau_0$ ,  $d \geq 1$ ,  $\alpha > 0$ .

The set of conditions  $i_1) - i_3)$  is called conditions (i).

For clarity, we note that all vector-functions are considered in the space  $C_{\tau,t}^{(k,\theta)}(R \times R^m)$ , and their dependence on the variables  $(\tau, t)$  or  $(\xi, \tau, t, h)$  is explicitly indicated whenever necessary.

Throughout the paper, the matrices  $A(\tau, t)$ ,  $K(\xi, \tau, t, h)$ ,  $Q(\xi, \tau_0, \tau, t, h)$ , as well as the operators  $\Phi, V, W$ , and  $R$  are assumed to be defined for all admissible values of their arguments, and their dependence on these variables will be omitted only when it does not lead to ambiguity.

### Construction of a multiperiod solution

Consider the linear inhomogeneous integro-differential equation

$$\begin{aligned} D_c u(\tau, t) &= A(\tau, t) u(\tau, t) + \\ &+ \int_{\tau}^{\tau+\theta} K(\xi, \tau, t, h(\xi, \tau, t)) u(\xi, h(\xi, \tau, t)) d\xi + f(\tau, t). \end{aligned} \tag{11}$$

Assume that the conditions (P\_1) are met if

$$f(\tau + \theta, t + \omega) = f(\tau, t) \in C_{\tau,t}^{(0,\theta)}(R \times R^m). \tag{12}$$

We will assume that equation (11) is uncritical (non-critical) for any vector-function  $f(\tau, t)$  of  $C_{\tau, t}^{(0, \theta)}(R \times R^m)$ .

Now let's start considering the problem for the system of integro-differential equations (11) with the initial condition

$$u|_{\tau=\tau_0} = \varphi(t), \quad (13)$$

where  $\varphi(t + \omega) = \varphi(t) \in C_t^{(\theta)}(R^m)$  is a well-known vector-function.

The function  $u(\tau, t)$  belonging to class  $C_{\tau, t}^{(1, \theta)}(R \times R^m)$  satisfying the integro-differential equation (11) and for  $\tau = \tau_0$  – the initial condition (13) is called the solution of the Cauchy problem for the inhomogeneous integro-differential equation (11).

Lemma 1. If the conditions (P),  $\varphi(t) \in C_t^{(\theta)}(R^m)$  are met, then there is a solution to the initial problem (1), (13) and it is given by the formula

$$\hat{u}(\tau, t) = \Phi(\tau_0, \tau, t)\varphi(h(\tau_0, \tau, t)), \quad \tau_0 \in R. \quad (14)$$

Proof. Based on (10), the formula (14) is represented as

$$\begin{aligned} \hat{u}(\tau, t) &= \mathbb{V}(\tau_0, \tau, t)\mathbb{W}(\tau_0, \tau, t)\varphi(h(\tau_0, \tau, t)) = \mathbb{V}(\tau_0, \tau, t)[\varphi(h(\tau_0, \tau, t)) + \\ &+ \int_{\tau_0}^{\tau} \int_{\eta}^{\eta+\theta} R(\xi, s, \eta, h(\eta, \tau, t), h(\xi, \tau, t)) \varphi(h(\tau_0, \tau, t)) d\eta d\xi]. \end{aligned} \quad (15)$$

In (15), we used representation (9) along the way. Taking into account the properties of the operator  $D_c, i_1$ ,  $D_c h(\tau_0, \tau, t) = 0$  and applying  $D_c$  to the relation (15), we have that  $\hat{u}(\tau, t)$  is the solution of the problem under consideration.

It follows from (14) that for  $\tau = \tau_0$ , condition (13) holds.

Q.E.D.

Lemma 2. If conditions (P) and (P\_1) are satisfied, then equation (11) admits a solution

$$\tilde{u}(\tau, t) = \int_{\tau_0}^{\tau} \Phi(\xi, \tau, t) f(\xi, h(\xi, \tau, t)) d\xi, \quad (16)$$

satisfying condition

$$\tilde{u}|_{\tau=\tau_0} = 0. \quad (17)$$

Proof. We have

$$\begin{aligned} &= f(\tau, t) + \int_{\tau_0}^{\tau} [D_c \Phi(\xi, \tau, t) + A(\tau, t)\Phi(\xi, \tau, t)] f(\xi, h(\xi, \tau, t)) d\xi = \\ &= f(\tau, t) + \int_{\tau_0}^{\tau} \mathbb{T}\Phi(\xi, \tau, t) f(\xi, h(\xi, \tau, t)) d\xi = f(\tau, t) + \mathbb{T}\tilde{u}(\tau, t), \end{aligned} \quad (18)$$

where  $(\mathbb{T}u)(\tau, t) = \int_{\tau}^{\tau+\theta} K(\xi, \tau, t, h(\xi, \tau, t))u(\xi, h(\xi, \tau, t)) d\xi$ .

Consequently, from (18) we have that the vector function  $\tilde{u}(\tau, t)$  satisfies equation (11).

The satisfiability (feasibility) of condition (17) is obvious.

It follows that  $\tilde{u}(\tau, t)$  is the solution of problem (11), (17).

Q.E.D.

Note that we can directly write the integral representation (16) in terms of the resolvent as follows

$$\begin{aligned} \tilde{u}(\tau, t) &= \int_{\tau_0}^{\tau} \mathbb{V}(\xi, \tau, t) [f(\xi, h(\xi, \tau, t)) + \\ &+ \int_{\xi}^{\xi+\theta} R(\xi', s, \eta, h(\eta, \tau, t), h(\xi, \tau, t)) f(\xi, h(\xi, \tau, t)) d\eta d\xi'] d\xi. \end{aligned} \quad (19)$$

Theorem 1. If the conditions (P),  $\varphi(t) \in C_t^{(\theta)}(R^m)$  are met, then there is a unique solution  $u(\tau, t)$  of the problem (11), (13) represented as

$$u(\tau, t) = \hat{u}(\tau, t) + \tilde{u}(\tau, t), \quad (20)$$

where  $\hat{u}(\tau, t)$  satisfies the integro-differential equation (1) and condition (13),  $\tilde{u}(\tau, t)$  – satisfies (11) and (17).

Proof. The proof of the theorem follows from Lemma 1 and Lemma 2.

Here the solutions  $\hat{u}(\tau, t)$  and  $\tilde{u}(\tau, t)$  are expressed in terms of (15) and (19), respectively. The uniqueness of the solution of the problem under consideration is proved by the method of the opposite.

The solution (20) of the Cauchy problem (11), (13) is represented as

$$u(\tau, t) = \Phi(\tau_0, \tau, t) \varphi(h(\tau_0, \tau, t)) + \int_{\tau_0}^{\tau} \Phi(\xi, \tau, t) f(\xi, h(\xi, \tau, t)) d\xi. \quad (21)$$

Using the necessary and sufficient periodicity condition

$$u(\tau_0 + \theta, t) = u(\tau_0, t) C_{\tau_0, \tau}^{(0, \theta)}(R \times R^m)$$

the method of successive approximations, and the method of complete mathematical induction, we obtain a function

$$\varphi^*(h(\tau_0, \tau, t)) = \int_{-\infty}^{\tau_0} \Phi(\xi, \tau_0, h(\tau_0, \tau, t)) f(\xi, h(\xi, \tau_0, t)) d\xi, \quad (22)$$

that satisfies the requirements of the initial condition (13).

Substituting (22) into (21), we obtain the corresponding solution to the Cauchy problem

$$u^*(\tau, t) = \int_{-\infty}^{\tau} \Phi(\xi, \tau, t) f(\xi, h(\xi, \tau, t)) d\xi. \quad (23)$$

Here we have used the group property. The convergence of the integral in (23) is provided by  $i_3$ ) and the estimate

$$\|u^*(\tau, t)\| \leq \int_{-\infty}^{\tau} \|\Phi(\xi, \tau, t)\| \|f(\xi, h(\xi, \tau, t))\| d\xi \leq \frac{d}{\alpha} \|f\|.$$

Problem. To establish the existence of a unique  $(\theta, \omega)$ – periodic solution of the integro-differential equation (11).

It is assumed that the resolving operator  $\Phi(\xi, \tau, t)$  is represented as

$$\Phi(\xi, \tau, t) = \Phi_{-}(\xi, \tau, t) + \Phi_{+}(\xi, \tau, t). \quad (24)$$

Here  $\Phi_{-}(\xi, \tau, t)$  and  $\Phi_{+}(\xi, \tau, t)$  are partial solutions of the vector-matrix integro-differential equation (1) with initial conditions

$$\begin{aligned} \Phi_{-}(\xi, \tau, t) &= E_{-}, \\ \Phi_{+}(\xi, \tau, t) &= E_{+}, \\ E &= E_{-} + E_{+} \end{aligned} \quad (25)$$

and with corresponding properties (i), as well as satisfying the estimates:

$$|\Phi_{-}(\xi, \tau, t)| \leq d e^{-\alpha(\tau-\xi)} \quad \text{при } \tau > \xi, \quad (26)$$

$$|\Phi_{+}(\xi, \tau, t)| \leq d e^{\alpha(\tau-\xi)}, \quad \tau < \xi, \quad (27)$$

where  $d \geq 1$ ,  $\alpha = \text{const} > 0$ .

Thus, splitting the resolving operator into the sum of matrices with properties (i), as well as satisfying conditions (26) and (27), leads to a system of integro-differential equations with the property of exponential dichotomy.

Let's start constructing a matrix function of the Green type.

$$\tilde{\Phi}(s, \tau, t) = \begin{cases} \Phi_-(s, \tau, t), & \tau > s, \\ -\Phi_+(s, \tau, t), & \tau < s. \end{cases} \quad (28)$$

Here are some properties of  $\tilde{\Phi}(s, \tau, t)$ :

$$i_4) D_c \tilde{\Phi}(s, \tau, t) = A(\tau, t) \tilde{\Phi}(s, \tau, t) + \int_{\tau}^{\tau+\theta} K(\xi, \tau, t, h(\xi, \tau, t)) \tilde{\Phi}(s, \tau, h(\xi, \tau, t)) d\xi, \quad \tau \neq s.$$

Due to the fact that  $\Phi_-(s, \tau, t)$  and  $\Phi_+(s, \tau, t)$  are matrix solutions of the integro-differential equation (1), the satisfiability (feasibility) of this condition follows.

$$i_5) \tilde{\Phi}(s - 0, \tau, t) - \tilde{\Phi}(s + 0, \tau, t) = E.$$

From (25), one can directly verify the feasibility of this property.

$$i_6) \tilde{\Phi}(s + \theta, \tau + \theta, t + \omega) = \tilde{\Phi}(s, \tau, t).$$

Note that  $E_-$ ,  $E_+$  and  $E$  do not depend on the variable  $t$ , as well as  $\Phi_-(s, \tau, t)$ ,  $\Phi_+(s, \tau, t)$  and  $\Phi(s, \tau, t)$  are multiperiodic matrix solutions, then the property  $i_6)$  follows from the above.

$$i_7) |\tilde{\Phi}(s, \tau, t)| \leq d e^{-\alpha|\tau-s|}, \quad d \geq 1, \quad \alpha > 0.$$

From estimates (26) and (27) we obtain the property  $i_7)$ .

It should be noted that the assumptions on boundedness, smoothness, and multiperiodicity of the matrices and kernels are sufficient for the application of the proposed method. These conditions ensure the existence of the matricant, the convergence of the resolvent series, and the boundedness of the resolving operator.

At the same time, these assumptions are not necessarily minimal. Their possible weakening, in particular with respect to smoothness or periodicity, requires additional investigation and may be considered in further studies.

Theorem 2. If the conditions (P), (P<sub>-1</sub>),  $k_0\theta + \sum_{j=1}^m k_j c_j \omega_j \neq 0$ ,  $\sum_{j=0}^m |k_j| \neq 0$ ,  $k_j \in Z$ , (24) and (25) are met, then the linear integro-differential equation (11) has a unique  $(\theta, \omega)$ -periodic solution representable in the form

$$u^{**}(\tau, t) = \int_{-\infty}^{+\infty} \tilde{\Phi}(\xi, \tau, t) f(\xi, h(\xi, \tau, t)) d\xi. \quad (29)$$

Proof. Using (28), we write (29) as follows

$$\begin{aligned} u^{**}(\tau, t) &= \int_{-\infty}^{\tau} \tilde{\Phi}(\xi, \tau, t) f(\xi, h(\xi, \tau, t)) d\xi + \\ &+ \int_{\tau}^{+\infty} \tilde{\Phi}(\xi, \tau, t) f(\xi, h(\xi, \tau, t)) d\xi = \\ &= \int_{-\infty}^{\tau} \Phi_-(\xi, \tau, t) f(\xi, h(\xi, \tau, t)) d\xi + \\ &+ \int_{\tau}^{+\infty} \Phi_+(\xi, \tau, t) f(\xi, h(\xi, \tau, t)) d\xi. \end{aligned}$$

Acting with the  $D_c$  operator, using the properties  $i_4)$  and  $i_5)$  we have

$$\begin{aligned} D_c u^{**}(\tau, t) &= [\tilde{\Phi}(\xi - 0, \tau, t) - \tilde{\Phi}(\xi + 0, \tau, t)] f(\tau, t) + \\ &+ \int_{-\infty}^{+\infty} D_c \tilde{\Phi}(\xi, \tau, t) f(\xi, h(\xi, \tau, t)) d\xi + \\ &+ \int_{-\infty}^{+\infty} \tilde{\Phi}(\xi, \tau, t) D_c f(\xi, h(\xi, \tau, t)) d\xi \end{aligned} \quad (30)$$

Further in (30) taking into account that for  $\tau \neq \xi$

$$\begin{aligned} D_c \Phi(\xi, \tau, t) &= [A(\tau, t)I + T]\Phi(\xi, \tau, t), \\ D_c f(\xi, h(\xi, \tau, t)) &= 0. \end{aligned} \quad (31)$$

where  $I$  – single operator;

$$(T\Phi)(\xi, \tau, t) = \int_{\tau}^{\tau+\theta} K(\xi, \tau, t, h(\xi, \tau, t))\Phi(\xi, \tau, t)d\xi$$

we have

$$D_c u^{**}(\tau, t) = f(\tau, t) + [A(\tau, t)I + T]u^{**}(\tau, t).$$

Relations (31) are used in deriving equation (30). Thus,  $u^{**}(\tau, t)$  satisfies equation (11) and is its solution.

The convergence of improper integrals is ensured by  $i_7)$  and the limitation of the function  $f(\tau, t)$ , and the convergence is uniform with respect to the vector  $t$  and the resolving operator  $\Phi(\xi, \tau, t)$  has the property of  $\omega$ -periodicity by  $t$ .

The  $(\theta, \omega)$ -periodicity of the solution (29) with respect to independent variables is derived from the multiperiodic property of the characteristic  $h(\xi, \tau, t)$  and the property (12) of the vector function  $f(\tau, t)$ . By shifting  $\tau$  by the period  $\theta$ , substituting  $\xi = s + \theta$ , and accounting for the periodicity of both  $f(\tau, t)$  and  $h(\xi, \tau, t)$ , the main requirement for the vector-function  $u^{**}(\tau, t)$  is satisfied:  $(\theta, \omega)$ -periodicity with respect to  $(\tau, t)$ . The uniqueness of the solution is proved by the method of the opposite.

Q.E.D.

Note some properties of the function  $u^{**}(\tau, t)$ .

1. The function  $u^{**}(\tau, t)$  satisfies equation (11).

2. It is a  $(\theta, \omega)$ -periodic function.

3.  $\|u^{**}(\tau, t)\| \leq \int_{-\infty}^{+\infty} \|\Phi(\xi, \tau, h(\xi, \tau, t))\| \|f(\xi, h(\xi, \tau, t))\| d\xi \leq \frac{\alpha}{\alpha} \|f\|$ ,

where  $\|f\| = \sup |f(\tau, t)|$ .

4. The solution  $u^{**}(\tau, t)$  is unique.

Remark. The obtained results can be applied to mathematical models describing processes with aftereffects, for example, in control theory and population dynamics, where the state of the system depends on its past values over a finite interval. In such models, the existence of multiperiodic solutions corresponds to stable oscillatory regimes determined by the periodic structure of the coefficients and the hereditary term.

## Results and discussion

In this paper, for the integro-differential system (1) with finite hereditary effects and the operator  $D_c$ , an integral representation of the solution manifold has been constructed using the method of periodic characteristics.

By means of the transformation (4), the original system is reduced to equation (5) with kernel  $Q$ , which preserves the properties of smoothness, boundedness, and multiperiodicity. Based on this, the matricant  $W(\tau_0, \tau, t)W(\tau_0, \tau, t)W(\tau_0, \tau, t)$  is defined as the solution of the integral equation (6), and its resolvent  $R$  is constructed via (7). The estimates obtained for the iterated kernels (8) ensure the convergence of the corresponding series and lead to the exponential bound for the resolvent. This, in turn, guarantees the boundedness of the resolving operator

$$\begin{aligned} \Phi(\tau_0, \tau, t) &= V(\tau_0, \tau, t)W(\tau_0, \tau, t), \quad \Phi(\tau_0, \tau, t) = V(\tau_0, \tau, t)W(\tau_0, \tau, t), \quad \Phi(\tau_0, \tau, t) \\ &= V(\tau_0, \tau, t)W(\tau_0, \tau, t), \end{aligned}$$

which satisfies conditions (i).

For the nonhomogeneous equation (11), the solution of the Cauchy problem (11), (13) is obtained in the form (21), combining the homogeneous component and the particular solution. This representation is expressed in terms of the resolving operator and reflects the influence of both the initial function and the right-hand side.

To construct a  $(\theta, \omega)$ -periodic solution, the resolving operator is decomposed as in (24) into  $\Phi_-$  and  $\Phi_+$ , satisfying the exponential estimates (26), (27), which correspond to an exponential dichotomy. On this basis, a Green-type matrix function  $\mathfrak{F}(s, \tau, t)$  is introduced by (28) and its main properties are established.

Using this function, it is proved that under conditions (P),  $(P_1)$  and the non-resonance condition, equation (11) admits a unique  $(\theta, \omega)$ -periodic solution represented by (29). The convergence of the improper integral in (29) follows from the exponential estimate of  $\mathfrak{F}$ , while multiplicity is ensured by the properties of the characteristic  $h(\xi, \tau, t)$  and the function  $f(\tau, t)$ .

Thus, the obtained results show that the existence of a resolving operator with properties (i) and its decomposition with exponential dichotomy provide sufficient conditions for the existence and uniqueness of a bounded multiperiodic solution of the integro-differential system with finite heredity.

## Conclusion

The significant role of the equations under consideration in modeling phenomena and processes associated with aftereffects is highlighted. For a partial integro-differential equation with a specialized differentiation operator, integral representations of the solution manifold have been derived. Several properties of iterated kernels and resolvents have been established, along with the corresponding estimates. Utilizing the necessary and sufficient conditions for periodicity, sufficient conditions for the existence of a bounded and unique multiperiodic solution across all independent variables on the periodic characteristics of a system of integro-differential equations with a specialized differentiation operator have been determined.

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## АҚЫРЛЫ-ЭРЕДИТАРЛЫ ИНТЕГРАЛДЫ-ДИФФЕРЕНЦИАЛДЫҚ ТЕНДЕУЛЕР ЖҮЙЕСІ ҮШІН КӨППЕРИОДТЫ ЕСЕПТІҢ ШЕШІМІН ЗЕРТТЕУ

### Андатпа

Интегралды-дифференциалдық теңдеулерді зерттеу қажеттілігі жаратылыстану мен техниканың бір-қатар салаларындағы көптеген қолданбалы есептермен негізделеді. Процестің немесе құбылыстың тарихын

сипаттайтын интегралды-дифференциалдық теңдеулерді зерттеу, интегралдық мүшенің қатысуын ескере отырып, В. Вольтерраның еңбектерінен бастау алады. «Тұқым қуалаушылықты», «салдарды» есепке алудың әртүрлі тәсілдері белгілі. Вольтерра типті интегралды-дифференциалдық теңдеулер жүйесінің көппериодты және квазипериодты шешімдерінің бар болуын анықтау әрдайым бірмәнді шешіле бермейтінін атап өтеміз. Арнайы дифференциалдау операторы бар дербес туындылы интегралды-дифференциалдық теңдеулер жүйесі қарастырылады. Периодтық сипаттамалар әдісін қолдану арқылы мұндай жүйелер шешімдерінің көпбейнесінің интегралдық көріністері салдарларымен бірге құрылды. Итерацияланған ядролар мен шешімдердің кейбір қасиеттері анықталып, сонымен қатар бағалаулар алынды. Көппериодты есептің Грин типті матрицалық функциясының бар болу шарттары және сәйкес бағалаулары бар интегралдық көрінісі табылды. Ақырлы эредитарлы интегралды-дифференциалдық жүйенің интегралдық көпбейнесі болатын көппериодты шешімнің бар болуы мен жалғыздығының жеткілікті шарттары алынды.

**Түйін сөздер:** матрицант, ядро, шешуші оператор, дихотомиялық, Грин типті функция.

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## ИССЛЕДОВАНИЕ МНОГОПЕРИОДИЧЕСКОЙ ЗАДАЧИ ДЛЯ КОНЕЧНО-ЭРЕДИТАРНОЙ СИСТЕМЫ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

### Аннотация

К проблеме необходимости исследования интегро-дифференциальных уравнений приводят многочисленные задачи приложения ряда областей естествознания и техники. Исследование интегро-дифференциальных уравнений, описывающие предысторию процесса или явления, начались с работ В. Вольтерра с учетом влияния интегрального члена. Известны различные подходы учета «наследственности», «последствия». Отметим, что установление существования многопериодических и квазипериодических решений системы интегро-дифференциальных уравнений типа Вольтерра не всегда однозначно разрешимо. Рассматривается система интегро-дифференциальных уравнений в частных производных со специальным оператором дифференцирования. Используя метод периодических характеристик, построены интегральные представления многообразия решений таких систем с последствием. Установлены некоторые свойства итерированных ядер и резольвенты, а также получены оценки. Найдены условия существования матричной функции типа Грина многопериодической задачи и интегральное представление с соответствующей оценкой. Получены достаточные условия существования и единственности многопериодического решения, которые являются интегральным многообразием интегро-дифференциальной системы с конечной эредитарностью.

**Ключевые слова:** матрицант, ядро, разрешающий оператор, дихотомичность, функция типа Грина.