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[1,2]**Tamabay D.,**
PhD student, ORCID ID: 0000-0001-8315-5849
e-mail: dtamabay@gmail.com
[1,2]**Temirbekov A.,**
PhD, Associate Professor, ORCID ID: 0000-0002-4157-2799
e-mail: almas_tem@mail.ru
[1,2]**Tleulesova A.,**
PhD, ORCID ID: 0000-0001-9280-1048
e-mail: aigerim1985_06@mail.ru
[1,2]***Mukhanova T.**
Bachelor's student, ORCID ID: 0009-0000-8522-3390
*e-mail: tomirismukhanova04@gmail.com

[1]Al-Farabi Kazakh National University, Almaty, Kazakhstan

[2]National Engineering Academy of the Republic of Kazakhstan, Almaty, Kazakhstan

ALPERT WAVELET-BASED GALERKIN METHOD FOR FIRST-KIND FREDHOLM INTEGRAL EQUATIONS

Abstract

This paper presents a numerical approach for solving Fredholm integral equations of the first kind using the Bubnov–Galerkin method with Alpert wavelet bases. These equations are well-known for being ill-posed, meaning that small changes in input data can lead to large deviations in the solution. Therefore, robust and accurate numerical methods are essential. The proposed method utilizes orthonormal and compactly supported Alpert wavelets, which offer excellent localization properties and yield well-conditioned, sparse system matrices when projecting the integral operator. This enhances numerical stability and reduces computational complexity. A series of computational experiments was carried out using various refinement levels and polynomial degrees. The accuracy of the method was evaluated by comparing approximate solutions to the exact analytical solution. The results demonstrate exceptionally small absolute errors, often approaching machine precision. Additionally, a comparative analysis with power polynomial bases confirms the superiority of the Alpert wavelet approach in terms of convergence and approximation quality. Overall, the method proves to be efficient, stable, and suitable for further extension to more complex integral equations, including multidimensional and noisy-data problems. This confirms the potential of Alpert wavelet-based Galerkin schemes as a reliable tool for the numerical treatment of inverse and ill-posed problems in applied sciences.

Keywords: first kind Fredholm Integral equation, Bubnov–Galerkin method, Alpert wavelet, ill-posedness, orthonormal basis.

Introduction

Mathematics and computational methods play a key role in solving a wide range of applied problems arising in engineering, physics, mechanics, atmospheric sciences, geophysics, geochemistry, geological exploration, medical imaging, remote sensing, military, and other high-tech disciplines. In many of these fields, a significant number of applied and theoretical problems can be reduced to the solution of Fredholm integral equations of the first kind [1–3].

It is worth noting that initial and boundary value problems for differential equations can be transformed into integral form, reducing the original problem to solving a single integral equation [4, 5]. In this case, computations are performed based on boundary data, which is especially important for problems with high dimensionality or complex domain geometries. However, a crucial theoretical and numerical aspect in dealing with Fredholm integral equations of the first kind is their ill-posedness: the solution strongly depends on the smoothness of the kernel and the right-hand side, and even small perturbations in the data may lead to large deviations in the solution. This makes it impossible to obtain an exact analytical solution in most cases and necessitates the use of stable and highly accurate numerical methods [6, 7].

The development of numerical methods for solving Fredholm integral equations of the first kind is particularly relevant due to the need to ensure convergence, stability, and approximation accuracy of the computed solution [8–10]. At the same time, an important aspect of theoretical analysis is the study of the conditions of existence and uniqueness of the solution, since for ill-posed problems, which include Fredholm integral equations of the first kind, such properties are not guaranteed without introducing additional restrictions or using special regularization methods [11–14]. In recent decades, various approaches have been proposed: direct numerical integration, regularization methods [15, 16] including the modified Tikhonov method [17], projection methods [18–20], smoothing techniques, GMRES-type algorithms, multilevel iterative schemes [21], as well as modern wavelet and wavelet-Galerkin methods [22–24], among others [25–27].

Wavelet-based methods have attracted increasing interest in recent years as a versatile tool for constructing adaptive bases with strong localization and high approximation order [28–29]. Since the first application of wavelets to the numerical solution of integral equations in 1991, their use has expanded considerably. Different types of wavelets–including Haar, Coifman, Legendre, and others – have demonstrated good performance in solving problems with various characteristics, confirming the high accuracy and stability of the approach [30–33]. Each wavelet type possesses specific properties–such as compact support, orthogonality, vanishing moments, symmetry, and smoothness – which determine its effectiveness in projection-based numerical methods.

Among the most promising directions in numerical modeling are wavelets constructed from classical orthogonal polynomials [30]. These constructions are actively used in spectral analysis, numerical solution of differential equations, signal and image processing, and high-accuracy approximations with controlled localization. In the context of projection methods for solving Fredholm integral equations of the first kind, the approximation properties of the basis functions are of particular importance. Effective choices include wavelet systems based on classical orthogonal polynomials such as Legendre, Chebyshev, Hermite, and Laguerre. These functions form orthonormal bases that satisfy requirements of localization, smoothness, and stability.

Legendre wavelets, orthogonal on the interval $[-1,1]$, have compact support and are well suited for implementing spectral-Galerkin schemes. Hermite wavelets, defined on $\mathbb{R}$, exhibit strong space-frequency localization and are applied in modeling solutions on the infinite domain. Chebyshev wavelets possess minimax properties and offer high approximation accuracy in the uniform norm, while Laguerre wavelets, defined on the half-line $[0,\infty)$, effectively approximate exponentially decaying functions [31–33].

The choice of a specific wavelet system as a basis in projection methods should be guided by both the geometry of the domain and the analytical properties of the solution and kernel of the integral equation. The basis properties of wavelets – such as orthonormality, locality, smoothness, and exponential decay – directly influence approximation quality, convergence, and the conditioning of the resulting algebraic systems.

Among all wavelets, Alpert wavelets occupy a special place – these are multilevel orthonormal bases constructed from Legendre polynomials [34]. They combine compact support, localized structure, and high smoothness, making them particularly effective within variational numerical methods. Their use in the Galerkin method for solving Fredholm integral equations of the first kind ensures high approximation accuracy and numerical stability, especially for smooth or locally inhomogeneous kernels. Due to the orthonormality of the basis functions, the computation of matrix elements is significantly simplified, reducing the conditioning of the linear system. Alpert wavelets thus represent a powerful tool in the spectrum of modern projection methods for numerical analysis of integral equations.

In this work, we present a numerical approach for solving Fredholm integral equations of the first kind based on the Bubnov–Galerkin method using Alpert wavelets as basis functions. Computational experiments are conducted with varying refinement levels and numbers of polynomial components, with visualizations of the obtained solutions. A comparative study is also carried out using standard basis functions in the form of power polynomials. The results demonstrate a high degree of approximation convergence of the numerical solution to the exact one, confirming the effectiveness of the chosen basis within the proposed numerical method.

Materials and methods

Fredholm integral equation of the first kind

Let $\varphi(x)$ be the unknown function, $f(x)$ the given function, and $K(x,t)$ the kernel. Let $\lambda \in \mathbb{R}$ be a parameter. The Fredholm integral equation of the first kind arises from the general equation when $\lambda = 0$:

$$\int_a^b K(x,t)\varphi(t)\,dt = f(x), \quad x \in [a,b]. \tag{1}$$

Here $K(x,t), f(x) \in L^2([a,b])$. The main feature of this equation is its ill-posedness. Specifically: solution may not be unique and can be extremely sensitive to small perturbations in the function $f(x)$.

Therefore, to solve such equations, it is necessary to apply regularizing, projection, or approximate methods.

Bubnov–Galerkin method

The Bubnov–Galerkin method is a projection method in which the solution $\varphi(t)$ is approximated by a linear combination of specially chosen basis functions:

$$\varphi(t) \approx \varphi_N(t) = \sum_{j=1}^{N} c_j \phi_j(t), \tag{2}$$

where $\{\phi_j(t)\}_{j=1}^{N}$ is a system of basis functions, and c_j are the coefficients to be determined.

Substituting this approximation into equation (1):

$$\int_a^b K(x,t)\left(\sum_{j=1}^{N} c_j \phi_j(t)\right)dt = f(x). \tag{3}$$

Interchanging the integral and the sum in (3) yields:

$$\sum_{j=1}^{N} c_j \int_a^b K(x,t)\phi_j(t)\,dt = f(x).$$

In the Bubnov–Galerkin method, the residual must be orthogonal to the basis functions. That is, for each $i = 1,2,\dots,N$, we require:

$$\sum_{j=1}^{N} c_j \int_a^b \int_a^b K(x,t)\phi_j(t)\phi_i(x)\,dt\,dx = \int_a^b f(x)\phi_i(x)\,dx. \tag{4}$$

We introduce notations:

$$A_{ij} = \int_a^b \int_a^b K(x,t)\phi_j(t)\phi_i(x)\,dt\,dx, \quad b_i = \int_a^b f(x)\phi_i(x)\,dx. \tag{5}$$

This leads to the linear system:

$$\sum_{j=1}^{N} A_{ij}c_j = b_i, \quad i = 1,2,\dots,N. \tag{6}$$

This is a linear algebraic system of the form $Ac = b$.

The effectiveness of the Bubnov–Galerkin method depends heavily on the choice of basis functions. Traditional bases include polynomials or trigonometric functions (e.g., Legendre or Chebyshev polynomials), which usually lead to dense matrices.

Alpert wavelets, however, are based on generalized Haar-like functions and form a multiscale family. When such bases are used, the matrix often becomes sparse, which greatly simplifies computations.

Alpert Wavelets

Alpert wavelets are defined using scaled, shifted, and rescaled versions of the first k Legendre polynomials on the interval $[0,1]$ via scaling functions $\{S^j(t)\}_{j=0}^{k-1}$. These functions have the following properties:

1. They are defined on the interval $[0,1]$;
2. They form an orthonormal basis

$$\int_0^1 S^j(t)S^i(t)dt = \delta_{ij}, \tag{7}$$

3. They span the space of polynomials of degree less than k.

Alpert wavelets are given by:

$$h_{j,m}^n(x) = 2^{m/2}h^j(2^m x - n), \quad n \in \mathbb{N}_0, \tag{8}$$

where m is the level of scaling and n is the shift parameter. These functions form a basis consisting of k wavelet functions localized on subintervals of length 2^{-m} in $[0,1]$.

For a fixed positive integer k and $m = 0,1,2,\dots$, the piecewise polynomial space S_m^k is defined as:

$$S_m^k = \{f : f|_{(2^{-m}n, 2^{-m}(n+1))} \in P_{k-1},\ n = 0,1,\dots,2^m - 1,\ f = 0 \text{elsewhere}\},$$

where P_{k-1} is the space of polynomials of degree less than k. The dimension of this space is $\dim(S_m^k) = 2^m k$, and the following nested structure holds:

$$S_0^k \subset S_1^k \subset \cdots \subset S_m^k \subset \cdots \tag{9}$$

For each m, define the orthogonal complement:

$$S_m^k \oplus R_m^k = S_{m+1}^k, \quad R_m^k \perp S_m^k, \quad \dim(R_m^k) = 2^m k. \tag{10}$$

By induction, we obtain:

$$S_m^k = S_0^k \oplus R_0^k \oplus R_1^k \oplus \cdots \oplus R_{m-1}^k. \tag{11}$$

Let $h_1,\dots,h_k : \mathbb{R} \to \mathbb{R}$ be an orthonormal basis of R_0^k. Since $R_0^k \perp S_0^k$, the first k moments of h_j vanish:

$$\int_0^1 h_j(x)x^i\,dx = 0, \quad i = 0,1,\dots,k-1. \tag{12}$$

To define $h_1, \dots, h_k$, we first construct functions $f_1, \dots, f_k: \mathbb{R} \to \mathbb{R}$ on $[0,1]$ with the following properties:

- Each f_i is a polynomial of degree at most $k-1$ on $[0,1]$;
- It extends to $(-1,0)$ as an even or odd function depending on the parity of $i+k-1$;
- They satisfy orthonormality and normalization:

$$\int_{-1}^{1} f_i(x) f_j(x)\, dx = \delta_{ij}, \quad i,j = 1, \dots, k, \tag{13}$$

- Each f_j satisfies moment vanishing conditions:

$$\int_{-1}^{1} f_j(x) x^i\, dx = 0, \quad i = 0,1, \dots, j+k-2. \tag{14}$$

- These functions f_j meet all four properties. Then,

$$h_i(x) = \sqrt{2} f_i(2x-1), \quad i = 1, \dots, k. \tag{15}$$

Thus,

$$R_0^k = \mathrm{span}\{h_i : i = 1, \dots, k\}. \tag{16}$$

In general,

$$R_m^k = \mathrm{span}\{h_{j,m}^n : h_{j,m}^n(x) = 2^{m/2} h_j(2^m x - n), j = 1, \dots, k, n = 0, \dots, 2^m - 1\}.$$

Alpert wavelets when $k = 2$:

$$h_1^0(t) = \begin{cases} -\sqrt{3}(4t-1), & if\ 0 \le t < \frac{1}{2}, \\ 0, & if\ \frac{1}{2} < t \le 1, \end{cases} \tag{17}$$

$$h_2^0(t) = \begin{cases} 0, & if\ 0 \le t < \frac{1}{2}, \\ \sqrt{3}(4t-3), & if\ \frac{1}{2} < t \le 1, \end{cases} \tag{18}$$

$$h_1^1(t) = \begin{cases} 6t-1, & if\ 0 \le t \le \frac{1}{2}, \\ 0, & if\ \frac{1}{2} < t \le 1, \end{cases} \tag{19}$$

$$h_2^1(t) = \begin{cases} 0, & if\ 0 \le t \le \frac{1}{2}, \\ 6t-5, & if\ \frac{1}{2} < t \le 1. \end{cases} \tag{20}$$

when $k = 3$:

$$h_1^0(t) = \begin{cases} -\frac{7}{3} + 24t - 40t^2, & if\ 0 \le t \le \frac{1}{2}, \\ 0, & if\ \frac{1}{2} < t \le 1, \end{cases} \tag{21}$$

$$h_2^0(t) = \begin{cases} 0, & if\ 0 \le t \le \frac{1}{2}, \\ \frac{55}{3} - 56t + 40t^2, & if\ \frac{1}{2} < t \le 1, \end{cases} \tag{22}$$

$$h_1^1(t) = \begin{cases} \sqrt{3}(1 - 14t + 30t^2), \ if \ 0 \leq t < \frac{1}{2} \\ 0, \ if \ \frac{1}{2} < t \leq 1 \end{cases} \tag{23}$$

$$h_2^1(t) = \begin{cases} 0, \ if \ 0 \leq t < -\frac{1}{2}, \\ \sqrt{3}(17 - 46t + 30t^2), \ if \ \frac{1}{2} < t \leq 1, \end{cases} \tag{24}$$

$$h_1^2(t) = \begin{cases} -\sqrt{5}\left(\frac{1}{3} + 6t - 16t^2\right), \ if \ 0 \leq t \leq \frac{1}{2}, \\ 0, \ if \ \frac{1}{2} < t \leq 1, \end{cases} \tag{25}$$

$$h_2^2(t) = \begin{cases} 0, \ if \ 0 \leq t \leq \frac{1}{2}, \\ \sqrt{5}\left(\frac{31}{3} - 26t + 16t^2\right), \ if \ \frac{1}{2} < t \leq 1. \end{cases} \tag{26}$$

Results and discussion

Test problem 1

Consider the following linear integral equation as a representative applied example of Fredholm integral equations of the first kind:

$$\int_0^1 \sqrt{x^2 + t^2}\, y(t)\, dt = \frac{(1+x^2)^{3/2} - x^3}{3}, \quad x \in [0,1].$$

It is known that the exact analytical solution to this equation is given by:

$$y(t) = t.$$

The following Table 1 presents the absolute error values between the exact analytical solution and the approximate solution obtained using the Alpert wavelet-based method at selected nodal points in the interval $t \in [0,1]$:

Table 1 – Absolute errors between the exact and approximate solutions at selected nodes

t	Exact solution	Approximate solution k=2, M=2	Absolute error k=2, M=2	Approximate solution k=2, M=3	Absolute error k=2, M=3
0.0	0.0	6.11715×10^{-10}	6.11715×10^{-10}	2.94681×10^{-5}	2.94681×10^{-5}
0.1	1.0×10^{-1}	1.00000×10^{-1}	2.88965×10^{-10}	1.00003×10^{-1}	3.08759×10^{-6}
0.2	2.0×10^{-1}	2.00000×10^{-1}	3.37850×10^{-11}	1.99989×10^{-1}	1.13691×10^{-5}
0.3	3.0×10^{-1}	2.99999×10^{-1}	3.56535×10^{-10}	2.99986×10^{-1}	1.39021×10^{-5}
0.4	4.0×10^{-1}	3.99999×10^{-1}	6.79285×10^{-10}	3.99995×10^{-1}	4.51122×10^{-6}
0.5	5.0×10^{-1}	4.99999×10^{-1}	8.12233×10^{-10}	4.99998×10^{-1}	2.10066×10^{-6}
0.6	6.0×10^{-1}	6.00000×10^{-1}	6.00417×10^{-10}	6.00003×10^{-1}	2.78820×10^{-6}
0.7	7.0×10^{-1}	7.00000×10^{-1}	3.88602×10^{-10}	7.00005×10^{-1}	4.66764×10^{-6}
0.8	8.0×10^{-1}	8.00000×10^{-1}	1.76788×10^{-10}	8.00004×10^{-1}	3.53763×10^{-6}
0.9	9.0×10^{-1}	9.00000×10^{-1}	3.50280×10^{-11}	9.00000×10^{-1}	6.01806×10^{-7}
1.0	1.0×10^{0}	1.00000×10^{0}	2.46844×10^{-10}	9.99992×10^{-1}	7.75068×10^{-6}

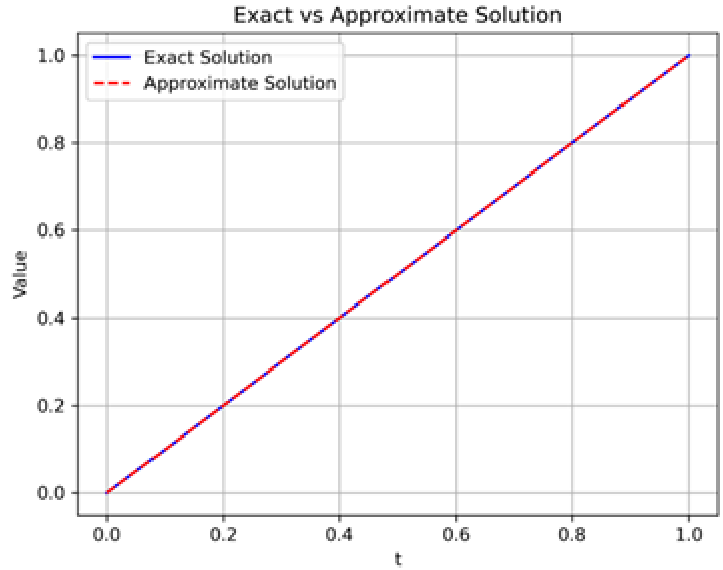

Figure 1 – Graph of exact solution and approximate solution of problem 1 in the case when $k = 2, M = 2$

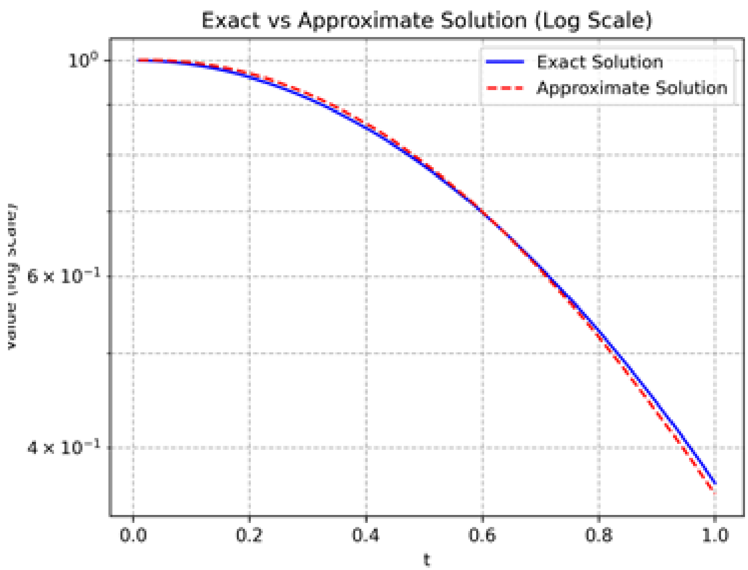

Figure 2 – Graph of exact solution and approximate solution of problem 1 in log scale in the case when $k = 2, M = 3$

Test problem 2

Consider the following linear integral equation as a representative applied example of Fredholm integral equations of the first kind:

$$\int_0^1 e^{xt} y(t)\, dt = \frac{e(x+1)-1}{1+x}, x \in [0,1].$$

It is known that the exact analytical solution to this equation is given by:

$$y(t) = e^t.$$

The following table presents the absolute error values between the exact analytical solution $y(t) = t$ and the approximate solution obtained using the Alpert wavelet-based method at selected nodal points in the interval $t \in [0,1]$:

Table 2 – Absolute errors between the exact and approximate solutions at selected nodes

t	Exact solution	Approximate solution k=2, M=2	Absolute error k=2, M=2	Approximate solution k=2, M=3	Absolute error k=2, M=3
0	1.00000×10^0	9.74063×10^{-1}	2.59365×10^{-2}	1.00032×10^0	3.19253×10^{-4}
1.0	1.10517×10^0	1.10487×10^0	2.99323×10^{-4}	1.10496×10^0	2.07005×10^{-4}
2.0	1.22140×10^0	1.23568×10^0	1.42770×10^{-2}	1.22163×10^0	2.29455×10^{-4}
3.0	1.34986×10^0	1.36649×10^0	1.66290×10^{-2}	1.35032×10^0	4.65347×10^{-4}
4.0	1.49182×10^0	1.49730×10^0	5.47124×10^{-3}	1.49104×10^0	7.84960×10^{-4}
5.0	1.64872×10^0	1.58505×10^0	6.36731×10^{-2}	1.65490×10^0	6.17632×10^{-3}
6.0	1.82212×10^0	1.80522×10^0	1.68980×10^{-2}	1.82271×10^0	5.88389×10^{-4}
7.0	2.01375×10^0	2.02539×10^0	1.16408×10^{-2}	2.01285×10^0	9.01444×10^{-4}
8.0	2.22554×10^0	2.24557×10^0	2.00252×10^{-2}	2.22533×10^0	2.11119×10^{-4}
9.0	2.45960×10^0	2.46574×10^0	6.13561×10^{-3}	2.46014×10^0	5.39715×10^{-4}
1.0	2.71828×10^0	2.68591×10^0	3.23705×10^{-2}	2.71729×10^0	9.91514×10^{-4}

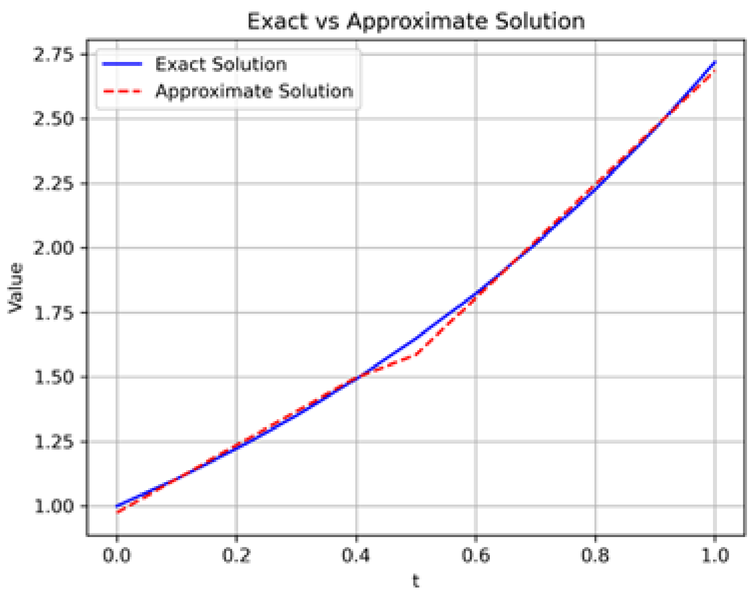

Figure 3 – Graph of exact solution and approximate solution
of problem 2 in the case when $k = 2, M = 2$

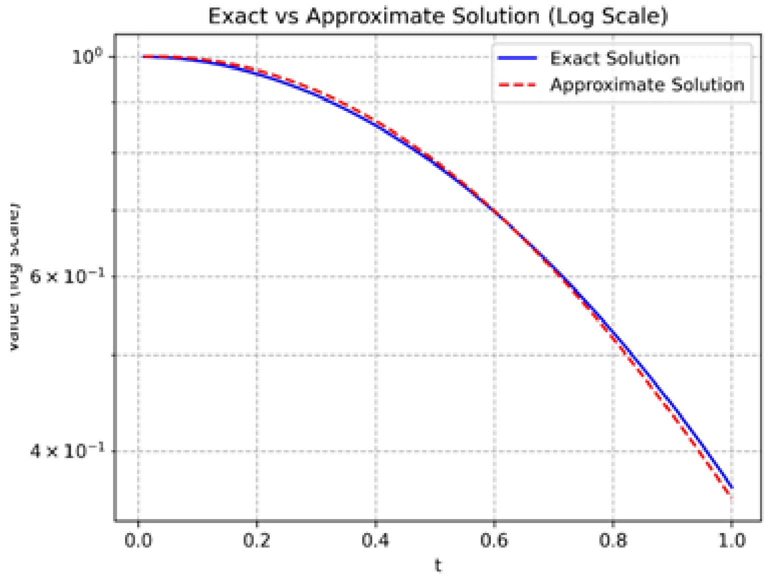

Figure 4 – Graph of exact solution and approximate solution of problem 2 in log scale in the case when $k = 2, M = 3$

When applying the Galerkin method to solve Fredholm integral equations of the first kind, the use of Alpert wavelets as basis functions presents both significant advantages and certain limitations, particularly in problems involving non-smooth kernels or solutions.

One of the main advantages of Alpert wavelets lies in their compact support, orthonormality, and piecewise-polynomial structure, which allow for efficient numerical implementation and sparse matrix representation. Their localized nature ensures that local features of the solution – such as peaks or sharp transitions – can be captured more accurately than with global basis functions like polynomials or trigonometric series. Additionally, the multi-resolution properties of wavelets provide a natural framework for adaptivity, enabling high accuracy with fewer basis functions when the solution is sufficiently smooth. These characteristics make Alpert wavelets particularly well-suited for ill-posed problems where stability and local refinement are essential [34].

However, the effectiveness of Alpert wavelets strongly depends on the smoothness of both the kernel $K(x,t)$ and the exact solution $u(t)$. If either contains discontinuities, singularities, or non-differentiable features, the approximation using a fixed number of wavelets becomes less accurate. In such cases, the method may require an excessive number of basis functions, which leads to the formation of an ill-conditioned linear system. This, in turn, increases the sensitivity of the solution to round-off and discretization errors, especially in the presence of small perturbations in the right-hand side function $f(x)$. Consequently, even minor numerical noise can result in significant deviations in the computed solution.

Furthermore, numerical integration used to compute the matrix coefficients in the Galerkin system may lose accuracy when the kernel is not smooth. The interaction between the wavelet basis and a non-smooth kernel can generate integrals with sharp variations that are difficult to evaluate precisely using standard quadrature techniques. Without adaptive integration or mesh refinement, these errors propagate into the system matrix and degrade the overall accuracy of the solution.

To mitigate these challenges, it is often necessary to introduce regularization techniques such as Tikhonov regularization, or to enhance the wavelet basis by including adaptive refinement or non-uniform scaling. These approaches help stabilize the numerical solution and improve approximation in the presence of singularities or discontinuities.

Conclusion

The numerical results obtained provide strong evidence for the accuracy and robustness of the Bubnov–Galerkin method when implemented with Alpert wavelet bases for solving first-kind Fredholm integral equations. A detailed analysis of the absolute errors between the exact analytical solution and the computed approximate solutions at uniformly distributed nodal points in the interval $t \in [0,1]$ demonstrates that the discrepancies are minimal – often approaching machine precision – thereby confirming the method's high numerical fidelity.

A key aspect contributing to the method's efficiency is the use of orthonormal and compactly supported Alpert wavelets. These basis functions ensure localized approximation and allow for the construction of sparse and well-conditioned system matrices upon projection of the integral operator. This, in turn, leads to enhanced numerical stability and reduced computational complexity, particularly beneficial when dealing with ill-posed problems.

Moreover, the structure of the resulting matrices facilitates scalable and efficient implementation in higher-dimensional problems. The localized wavelet approach not only improves accuracy but also enables selective resolution refinement, which is especially valuable in practical applications involving singularities, discontinuities, or noisy data.

The versatility of the proposed method is noteworthy. It exhibits strong potential for extension to more complex and realistic scenarios, including integral equations with discontinuous kernels, noisy right-hand sides, and multidimensional domains commonly encountered in physical, engineering, and geophysical modeling. The method remains stable and accurate under increased problem complexity and dimensionality, making it a promising computational tool for modern inverse and direct problems.

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[1,2]Тамабай Д.,
докторант, ORCID ID: 0000-0001-8315-5849
e-mail: dtamabay@gmail.com
[1,2]Темирбеков А.,
PhD, қауымдастырылған профессор, ORCID ID: 0000-0002-4157-2799
e-mail: almas_tem@mail.ru
[1,2]Тлеулесова А.,
PhD, ORCID ID: 0000-0001-9280-1048
e-mail: aigerim1985_06@mail.ru
[1,2]*Муханова Т.,
студент, ORCID ID: 0009-0000-8522-3390
*e-mail: tomirismukhanova04@gmail.com

[1]Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы қ., Казахстан
[2]ҚР Ұлттық инженерлік академиясы, Алматы қ., Казахстан

АЛЬПЕРТ ВЕЙВЛЕТ БАЗИСТЕРІН ҚОЛДАНА ОТЫРЫП ФРЕДГОЛЬМНІҢ БІРІНШІ ТҮРДЕГІ ИНТЕГРАЛДЫҚ ТЕҢДЕУІН БУБНОВ–ГАЛЕРКИН ӘДІСІМЕН САНДЫҚ ШЕШУ

Аңдатпа

Бұл мақалада Фредгольмнің бірінші түрдегі интегралдық теңдеулерін Бубнов–Галеркин әдісімен шешудің сандық тәсілі ұсынылады, мұнда базистік функциялар ретінде Альперт вейвлеттері пайдаланылады. Аталған теңдеулер қойылымының дұрыс еместігімен (иллю-пост) ерекшеленеді, яғни кіріс деректеріндегі шағын өзгерістер шешімге айтарлықтай әсер етуі мүмкін. Сондықтан мұндай теңдеулерді шешу үшін орнықты және дәл сандық әдістер қажет. Ұсынылып отырған әдісте орто-нормаланған және компакты тірегі бар Альперт вейвлеттері қолданылады, олар локализация қасиеттерімен ерекшеленеді және интегралдық операторды проекциялағанда жақсы жағдайланған, сиретілген (разреженные) матрицалар береді. Бұл өз кезегінде сандық орнықтылықты арттырып, есептеу күрделілігін азайтады. Әртүрлі тор тығыздықтары мен полиномдық дәрежелер үшін сандық эксперименттер жүргізілді. Әдістің дәлдігі нақты аналитикалық шешіммен салыстыру арқылы бағаланды. Нәтижелер абсолюттік қателіктердің өте кіші екенін, кейбір жағдайларда машиналық дәлдікке жақындайтынын көрсетті. Сонымен қатар, полиномдық базистермен салыстырмалы талдау Альперт вейвлеттеріне негізделген тәсілдің жуықтау сапасы мен жинақтылық жылдамдығы жағынан басымдығын растады. Жалпы алғанда, әдіс тиімділігімен, орнықтылығымен және көпөлшемді немесе шуды деректер жағдайларына кеңейту мүмкіндігімен ерекшеленеді. Бұл Альперт вейвлеттері негізінде құрастырылған Галеркин сұлбаларының қолданбалы ғылымдардағы дұрыс қойылмаған және кері есептерді сандық шешуге арналған сенімді құрал екенін көрсетеді.

Тірек сөздер: Фредгольмнің бірінші түрдегі интегралдық теңдеуі, Бубнов–Галеркин әдісі, Альперт вейвлеттері, дұрыс қойылмаған есеп, орто-нормаланған базис.

[1,2]Тамабай Д.,
докторант, ORCID ID: 0000-0001-8315-5849
e-mail: dtamabay@gmail.com
[1,2]Темирбеков А.,
PhD, ассоциированный профессор, ORCID ID: 0000-0002-4157-2799
e-mail: almas_tem@mail.ru
[1,2]Тлеулесова А.,
PhD, ORCID ID: 0000-0001-9280-1048
e-mail: aigerim1985_06@mail.ru
[1,2]*Муханова Т.,
студент, ORCID ID: 0009-0000-8522-3390
*e-mail: tomirismukhanova04@gmail.com

[1]Казахский национальный университет им. аль-Фараби, г. Алматы, Казахстан
[2]Национальная инженерная академия РК, г. Алматы, Казахстан

ЧИСЛЕННОЕ РЕШЕНИЕ ИНТЕГРАЛЬНОГО УРАВНЕНИЯ ФРЕДГОЛЬМА ПЕРВОГО РОДА МЕТОДОМ БУБНОВА–ГАЛЁРКИНА С ИСПОЛЬЗОВАНИЕМ БАЗИСОВ ВЕЙВЛЕТОВ АЛЬПЕРТА

Аннотация

В данной статье представлен численный подход к решению интегральных уравнений Фредгольма первого рода с применением метода Бубнова–Галёркина и базисных функций в виде вейвлетов Альперта. Эти уравнения известны своей некорректностью: малейшие изменения во входных данных могут вызывать существенные отклонения в решении. В связи с этим необходимы устойчивые и точные численные методы. Предлагаемый метод использует ортонормированные и компактно поддержанные вейвлеты Альперта, обладающие отличными локализационными свойствами. Они обеспечивают хорошо обусловленные и разреженные матрицы системы при проецировании интегрального оператора, что повышает численную устойчивость и снижает вычислительную сложность. Серия численных экспериментов была проведена с различными уровнями уточнения и степенями полиномов. Точность метода оценивалась путем сравнения приближенных решений с точным аналитическим решением. Результаты показали исключительно малые абсолютные ошибки, зачастую близкие к машинной точности. Кроме того, сравнительный анализ с базисами из степенных полиномов подтвердил превосходство подхода, основанного на вейвлетах Альперта, как по скорости сходимости, так и по качеству аппроксимации. В целом метод продемонстрировал эффективность, устойчивость и пригодность к дальнейшему расширению на более сложные классы интегральных уравнений, включая многомерные задачи и задачи с зашумленными данными. Это подтверждает потенциал вейвлетных схем Галёркина с базисами Альперта как надежного инструмента для численного решения некорректных и обратных задач в прикладных науках.

Ключевые слова: интегральное уравнение Фредгольма первого рода, метод Бубнова–Галёркина, вейвлет Альперта, некорректность задачи, ортонормированный базис.

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