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A NOTE ON THE STRUCTURE OF MINIMAL DARK CEERS

Abstract

The structure of computably enumerable equivalence relations under computable reducibility (commonly referred to as ceers) has been actively developed over the past 25 years. A comprehensive survey by Andrews and Sorbi presented numerous structural properties of ceers, most notably investigating the existence of joins and meets in the degree structure of ceers. They divided the structure into two definable parts: dark ceers (ceers without an effective transversal) and light ceers (ceers with an effective transversal). They also showed the existence of an infinite number of minimal dark ceers (modulo equivalence relations with finitely many classes). Minimal dark ceers exhibit the distinctive property that every pair of classes is computably inseparable. Furthermore, the classes of weakly precomplete equivalence relations (i.e. those that lack a computable diagonal functions) are also computably inseparable. In this context, a natural question arises: do minimal dark equivalence relations exist that are not weakly precomplete? This paper provides an affirmative answer to this question. Moreover, we establish the existence of an infinite family of non-weakly precomplete minimal dark ceers that avoids lower cone of a given non-universal ceer. We denote by $\mathbf{FC}$ the set of ceers consisting of only finite classes. Andrews, Schweber, Sorbi showed the existence of dark $\mathbf{FC}$ equivalences. In this paper, we prove that over any dark $\mathbf{FC}$ ceer, there exists an infinite antichain of dark $\mathbf{FC}$ ceers.

Key words: Equivalence relation, computably enumerable equivalence relation, computable reducibility, weakly precomplete equivalence relation.

Introduction

The paper studies properties of computable reducibility for computably enumerable binary relations on the set of natural numbers ω. Suppose R and S are binary relations on ω. We say that R is computably reducible to S (denoted by $R \leq_c S$) if and only if there exists a computable function $\mathbf{f}$ such that for every $x, y \in \omega$

$$(x, y) \in R \Leftrightarrow \big(f(x), f(y)\big) \in S.$$

Relations R and S are said to be computably equivalent if and only if $R \leq_c S$ and $S \leq_c R$. Throughout this paper, we will be working with computably enumerable equivalence relations, or ceers for short. A ceer U is called universal if $E \leq_c U$ for every ceer E. For more details on $\leq_c$-reducibility in the context of equivalence relations, see [1–13, 16, 19, 21].

An equivalence relation E is called precomplete if there exists a total computable function $f(e,x)$ such that for all $e, x \in \omega$, whenever $\varphi_e(x)$ is defined, it holds that $\varphi_e(x)\ E\ f(e,x)$. According to Ershov's Fixed-Point Theorem, an equivalence relation E is precomplete if and only if there exists a total computable function $f(x)$ such that for every $e \in \omega$, if $\varphi_e(f(e))$ is defined, then $\varphi_e(f(e))\ E\ f(e)$ [14].

Badaev weakened the notion of precompleteness by introducing weakly precomplete equivalences [15]: an equivalence relation E is called weakly precomplete if there exists a partial computable function fix such that the following holds for each $e \in \omega$:

$$(\varphi_e \text{ } is \text{ } total) \Rightarrow \left[fix(e)\downarrow \& \left(\varphi_e(fix(e))\ E\ fix(e)\right)\right].$$

Later, Badaev and Sorbi provided a criterion for weakly precompleteness using so-called diagonal functions [16]. A total function d is called a diagonal function for an equivalence relation E, if for every x, $(d(x), x) \notin E$.

Proposition 1.1. Let E be a ceer. The following statements are equivalent:
1. E is weakly precomplete;
2. $(\forall e)[\varphi_e \text{ } is \text{ } total \Rightarrow (\exists n)[\varphi_e(n)\ E\ n]]$;
3. E has no computable diagonal function.

Properties of diagonal functions for ceers were studied extensively in [17].

A ceer E is dark if it is incomparable with $Id = \{(x,x) | x \in \omega\}$. A ceer E is called light if $Id \leq_c E$. Equivalently, E is called light if and only if there is an effective transversal, i.e. there exists a c.e. set W whose elements are pairwise non- E-equivalent [18].

Andrews and Sorbi showed that, unlike the light ceers (where Id is the least light ceer), there is no least dark ceer [18]. In particular, they proved the existence of infinitely many incomparable minimal dark ceers (see Theorem 3.3 in [18]).

We summarize some basic facts about minimal dark ceers based on Theorem 3.3 in [18].

Observation 1.2. Let E be a minimal dark ceer. Then
1. every pair of equivalence classes of E is computably inseparable;
2. every equivalence class of E is non-computable;
3. every equivalence class of E is infinite.

These statements also apply to weakly precomplete ceers.

A fundamental property of computably enumerable equivalence relations is the finiteness of their equivalence classes. A ceer E is called $\mathbf{FC}$ (standing for finite classes) if all its equivalence classes are finite [19]. A stronger notion is k-boundedness, meaning that every equivalence class of E contains at most k elements for some $k \in \omega$ with $k > 0$. If such a bound exists for some k, E is simply called bounded [6]. An important structural property follows: every bounded ceer is light (see Theorem 6.2 (1) in [19]).

Theorem 1.3 ([20]). For every non-universal positive preorder R, there exists a weakly precomplete minimal positive preorder P, which is not reducible to R.

Corollary 1.4 ([20]). For every positive non-universal preorder R, there exist infinitely many pairwise incomparable weakly precomplete minimal positive preorders, none of which is reducible to R.

It can be observed that Theorem 1.3 and Corollary 1.4 remain valid if we replace the term "positive preorder" with "positive equivalence relation". This naturally leads to a question of whether minimal dark equivalences exist that are not weakly precomplete. Theorem 3.8, the main result of this paper, gives an affirmative answer for this question.

The remaining of the paper is organized as follows. In the section Materials and Methods we explain the finite injury priority method, which serves as the main tool in our proofs. In the section

Results and Discussion, we establish an existence of non-weakly precomplete minimal dark ceers and prove a result analogous to Corollary 1.4. Furthermore, we show that over any dark $\mathbf{FC}$ ceer, there exist infinitely many incomparable dark $\mathbf{FC}$ ceers (Theorem 3.1).

Materials and methods

Throughout the main section, we will use a standard technique from classical computability theory called finite injury priority method.

Suppose, we want to build a constructive object (c.e. set, equivalence relation, linear order etc.) with certain properties. To achieve this, we will play against an effective listing (or a numbering) of objects (such as c.e. sets, partial computable functions etc.) trying to satisfy specific requirements that ensure the desired properties.

The fulfillment of each such requirement is dictated by strategies. Each strategy R_e typically has a witness or representative x_e that witnesses satisfactory outcome of each requirement. It might happen so that a requirement may fail to be met under the chosen representative, in this case we may restart the strategy by choosing a new witness. This restarting process is called initialization.

For example, suppose we want to construct a set A with certain properties. Let $\{R_0, R_1, R_2, \dots\}$ be the set of all requirements that must be satisfied for such set A to exist. Assume that R_{2i} wants to enumerate elements into A, while R_{2j+1} aims to prevent certain elements from entering A. Clearly, even-numbered strategies will conflict with odd-numbered strategies and vice versa.

This is where the priority method comes into play. It allows us to order requirements in such a way that conflicts between different strategies cannot be completely avoided but are instead organized so that lower-priority requirements respect the constraints imposed by higher-priority ones. That is, we define a priority ordering:

$$R_0 < R_1 < R_2 < R_3 < \cdots$$

A strategy R_i can ignore constraints imposed by any R_j with $j > i$, but R_j must select its actions so that the constraints set by each R_i are not violated.

The number of possible outcomes for a strategy may vary depending on the problem, but the main ones are act and wait. We say that the strategy for R_e has the outcome act if there is a witness x_e that meets R_e. Otherwise, we say that the strategy has the outcome wait. Of course, at the stage s, we may not yet know what the ultimate outcome of a given strategy will be.

The process works as follows: suppose we want to enumerate a number x into A to satisfy R_{20}, but earlier, we observed that R_{13} put a restrain on enumerating x into A. As a result, R_{20} must look for another element that respects the constraint imposed by R_{13} – meaning that at stage s, R_{20} is in the wait state. However, higher-priority strategies, such as those for $R_0, R_2, \dots, R_{12}$, may injure the constraint imposed by R_{13} by forcing x into A, thereby switching the strategy to an act outcome. In this case, we will need to reinitialize R_{13} at most seven times. After that, the new constraint imposed by R_{13} on a different number x' will be respected by the other even-numbered strategies. When each strategy experiences only a finite number of injuries, we say that the object is constructed using the finite injury priority method.

One way to conceptualize finite injury arguments is through tree of strategies. We introduce this notion informally here and refer a curious reader to for all formal details [22]. A tree of strategies T is a full binary tree (where the branching factor depends on the number of possible outcomes of the strategies), with nodes corresponding to requirements R_i (see Figure 1). A path through T is a string $\sigma \in \{a,w\}^{\omega}$. Here, a and w correspond to the act and wait outcomes, respectively. A true path of T is a path TP such that every requirement along this path is satisfied.

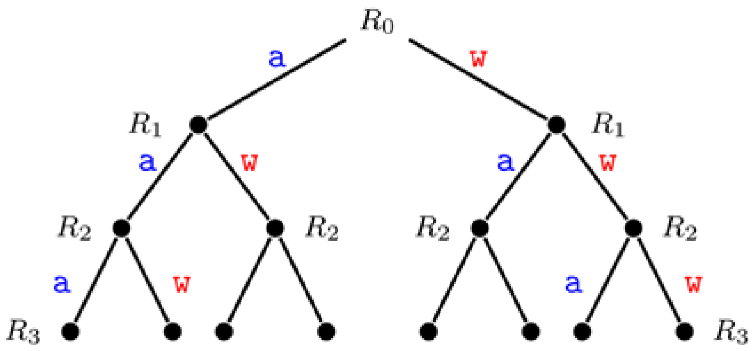

Figure 1 – Tree of strategies T

Since our construction is done by stages, at each stage s, we define an approximation of TP as a unique string $TP_s \in \{a,w\}^{<\omega}$ such that $|TP_s| = s$. For example, suppose that at some stage $s > 0$, the strategy R_0 is still waiting for its witness to act, R_1 has already placed a restraint on some number, and R_2 has acted according to its strategy. In this case, TP_s is currently defined as a green path in Figure 2. At the next stage, $s + 1$, the strategy for R_0 may act on its witness, but this could injure R_1 and R_2, requiring us to initialize them. After initialization, strategies for R_1 and R_2 may no longer be satisfied. This situation is illustrated in Figure 3 below. Additionally, note that $(\forall t > s)[TP_t(0) = TP(0)]$.

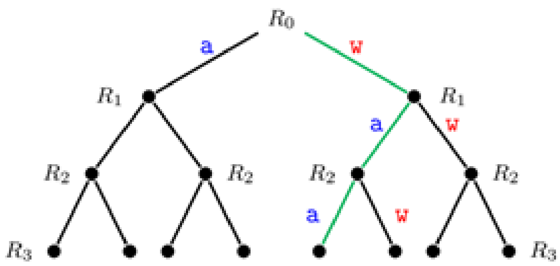

Figure 2 – Approximation of TP at stage s

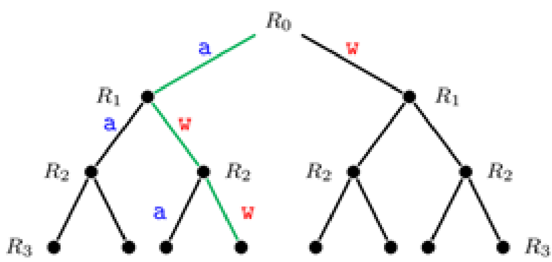

Figure 3 – Approximation of TP at stage $s + 1$

As seen in the pictures above, some requirements have multiple copies as backups. This is done to ensure that every requirement is eventually satisfied. Note that each strategy can remain in the waiting state for a long time unless it acts. In the latter case, if no higher-priority strategies injure it, the requirement will never switch back to the waiting state, thereby stabilizing some initial segment of TP. Since higher-priority strategies either never act or act only once, each $TP_s(i)$ (i^{th} element of the string TP_s) changes only finitely many times.

Results and discussion

It can be shown that $\mathbf{FC}$ dark ceers exist (see, e.g., Example 3.2 in [21]). We show that the upper cone of any such ceer contains an infinite antichain of $\mathbf{FC}$ dark ceers.

Theorem 3.1. Let F be any $\mathbf{FC}$ dark ceer. Then there exist infinitely many incomparable $\mathbf{FC}$ dark ceers $(E_l)_{l\in\omega}$ such that $F < E_l$ for all $l \in \omega$.

Proof. We construct a sequence of ceers $(E_l)_{l\in\omega}$ such that in each E_l the elements of $W = \omega^{[0]} = \{\langle 0, x\rangle | x \in \omega\}$ are reserved for coding F. That is, in E_l we E_l-collapse $\langle 0, x\rangle$ and $\langle 0, y\rangle$ if and only if $(x, y) \in F$. We satisfy the following requirements:

$Q_e^{k,l}$: If $k \neq l$ then φ_e is not a reduction from E_k to E_l.

P_e^l: φ_e is not a reduction from Id to E_l.

F_e^l: The E_l-class of e is finite.

We fix a computable order on the requirements of order type ω. We say that a requirement K has higher priority than a requirement K' if K strictly precedes K' in the order defined above (in this case, K' has lower priority than K). Coding F on the set $W = \omega^{[0]}$ in each E_l will ensure reduction $F \leq E_l$. The set W will grow in the sense that we will redefine W each time when some element $a \in W$ gets E_l-collapsed with $b \notin W$. We allow strategies to set two types of restraints for strategies of lower priority: $R(a, b, l)$ meaning that a can not be E_l-collapsed with b; and $R(W; a, l)$ meaning that a can not be E_l-collapsed with any element from W.

Strategy for $Q_e^{k,l}$. Informally, we will try to diagonalize directly when possible, if not we will code a universal ceer into E_l. Fix a universal ceer U with its computable approximation $\{U_s\}_{s\in\omega}$. We start by setting a counter $m = 0$ then proceed by choosing a pair of fresh elements x_0 and x_1 taken from outside of W part of E_k and place a restraint $R(x_0, x_1, k)$. At each later stage, we continue to look for fresh elements x_m and placing restraints $R(x_i, x_m, k)$ while growing our counter m for every $i < m$. If the function φ_e fails to halt on at least one of the chosen witnesses, that is, if we wait for infinity, it means that the function φ_e is not a reduction at all, and we have won.

Formal description of the strategy:

1) Set $m := 1$;

2) Choose new witnesses x_i for $i \leq m$ and for any $i, j \leq m$ with $i \neq j$ place restraints $R(x_i, x_j, k), R(W; x_i, k)$;

3) Wait for $\varphi_e(x_i)$ to converge for all $i \leq m$;

4) Assume $\varphi_e(x_i) = y_i$ for all $i \leq m$.

(a) If there are y_i and y_j such that $(y_i, y_j) \notin E_l$ and both ys are from outside of W in E_l, then we E_k-collapse x_i and x_j and place restraints $R(y_i, y_j, l), R(W; y_i, l)$ and $R(W; y_j, l)$.

(b) If $y_i\ E_l\ y_j$ and $(x_i, x_j) \notin E_k$ for some $i, j \leq m$, then we keep restraints $R(x_i, x_j, k), R(W; x_i, k)$ and $R(W; x_j, k)$.

(c) If there are $i,j \leq m$ such that $y_i \in W, y_j \notin W$, and $(y_i, y_j) \notin E_l$, then we E_k-collapse x_i and x_j, and set the restraint $R(y_i, y_j, l)$.

(d) If every $y_i \in W$ and for every $i, j \leq m$ we have $x_i\ E_k\ x_j \Leftrightarrow y_i\ E_l\ y_j$ such that $r(p_i)\ F\ r(p_j)$ where

$$p_t = \mu q[q = \langle 0, y\rangle \ \&\ q\ E_l\ y_t], \tag{1}$$

then we E_k-collapse every pair $(x_i, x_j) \Leftrightarrow i\ U\ j$, increase our counter m by 1, and proceed to define x_{m+1}, i.e. we set $m := m + 1$, choose a new witness x_m, and go to Step 3.

The outcomes are either wait at Step 3 waiting for φ_e to converge, or act when φ_e appears to converge everywhere, while at the same time not being closely tied to W part (Steps 4(a), 4(b) and 4(c)), in which case we can diagonalize. Alternatively, φ_e might tend to mimic F; if this happens infinitely often, we will eventually obtain a reduction from U to F via the function $f(t) = r(p_t)$, which contradicts the fact that F is dark.

Strategy for P_e^l.

1) Set $m := 1$;

2) Wait for $\varphi_e(i)$ to converge for every $i \leq m$;

3) Assume $y_i = \varphi_e(i)$ for all $i \leq m$.

(a) If there exist y_i and y_j $(i \neq j)$ such that $y_i\ E_l\ y_j$, then the requirement is satisfied.

(b) If all y_i s are pairwise non-E_l-equivalent and there exist y_i and y_i $(i \neq j)$ such that they do not have restraints from higher priority strategies, then we E_l-collapse (y_i, y_j) and say that the requirement is satisfied.

(c) If all y_is are pairwise non-E_l-equivalent and have restrictions posed by higher priority strategies, then set $m := m + 1$, and go to Step 2.

The outcomes are either wait at Step 2, or act at Step 3. We diagonalize directly in Steps 3(a) and 3(b). If m continues to grow infinitely often, then it means that infinitely many y_is have restraints set by higher priority requirements. However, this is impossible because there are only finitely many higher priority requirements placing no more than finite number of restraints each (see Lemma 3.3).

Strategy for F_e^l. This strategy restrains the E_l-class of e from collapsing with any other E_l-class by lower priority requirement. Since there are only finitely many requirements of higher priority, E_l-class of e will be finite. We will denote such restraints by $R(e, l)$.

Construction

The construction is by stages. More specifically, we will construct computable approximations $\{E_{l,s}\}_{s\in\omega}$ of E_l for all $l \in \omega$. We say that a requirement $K \in \{Q_e^{k,l}, P_e^l : e, k, l \in \omega\}$ is initialized meaning that it chooses (or looks for) a new set of parameters. We say that a requirement $K \in \{Q_e^{k,l}, P_e^l : e, k, l \in \omega\}$ requires attention at stage s, if K is initialized at stage s, or K has not taken any action since its last initialization, but is now ready to take action. Keep in mind that each requirement must deal with restraints of the form $R(a, b, l), R(W; a, l)$ and $R(a, l)$ imposed by higher priority strategies. If $K = F_e^l$, then it sets up the restraint $R(e, l)$ once and it will never be canceled.

Stage 0. Initialize all Q and P requirements and define $E_{l,0} = Id$.

Stage $s + 1$. Code F_{s+1} in every E_l with $l < s + 1$, i.e. E_l-collapse $\langle 0, x\rangle$ and $\langle 0, y\rangle \Leftrightarrow x\ F_{s+1}\ y$. Consider the least K requirement that requires attention at the current stage.

If K is a P-requirement, then take K-action. If K is a F-requirement then place a corresponding restraint. Otherwise,

- if K is initialized, then choose new parameters for it as described in the strategy for K;
- if K is not initialized, then take K-action as described.

After K has acted, initialize all requirements $K' > K$.

End of stage. Define $E_{l,s+1} := E_{l,s} \cup \{\text{all pairs which have been } E_l\text{-collapsed at stage } s + 1\}$. Proceed to the next stage.

Verification

We show that each requirement will eventually be satisfied by proving Lemmas 3.2, 3.3, 3.4 below.

Lemma 3.2. Every $Q_e^{k,l}$ requirement is satisfied.

Proof. Note that every requirement $Q_e^{k,l}$ either hangs in the waiting state forever at Step 3 of the strategy or eventually acts at Step 4(a) or 4(b) or 4(c). After each action, $Q_e^{k,l}$ can be injured only finitely many times by higher priority strategies. Therefore, it will act at most finitely many times. Thus, $Q_e^{k,l}$ is eventually satisfied. □

Lemma 3.3. If φ_e is total, then the outcome for P_e^l is act.

Proof. By way of contradiction, suppose that φ_e is total and we don't see Steps 3(a) and 3(b) infinitely often. Note that we won't get stuck at Step 2 because φ_e is total. Therefore, we must be stuck at Step 3 infinitely often. Since we don't see 3(a), it follows that $(\forall i \neq j)[(\varphi_e(y_i), \varphi_e(y_j)) \notin E_l]$. Furthermore, since we don't see 3(b) either for all $i \neq j$ pairs (y_i, y_j) have restraints set by higher priority strategies, or both y_i and y_j are in W. Since only finitely many restraints are set by higher priority strategies, there exists some $N \in \omega$ such that for all $k > N$ we have $y_k \in W$. Let's introduce a function $p_N(x)$ as a modification of the function defined in Eq. (1) as follows:

$p_N(x) = p_k$, with $y_k = x + y_N$.

It is not difficult to verify that the function $g(x) = r(p_N(x))$ computably reduces Id to F, contradicting to the fact that F is dark. □

Lemma 3.4. Every class of each E_l, $l \in \omega$ is finite.

Proof. The statement is equivalent to saying that every F_e^l-requirement is satisfied. By Lemma 3.2 and Lemma 3.3 every $Q_i^{k,l}$ and P_j^l requirements act finitely often, where $i, j \leq e$. Hence, the class $[e]_{E_l}$ grows only finitely many times. □

This completes the proof of Theorem 3.1. □

Next, we present some facts about $\mathbf{FC}$ dark ceers related to minimality.

Corollary 3.5. No minimal dark ceer is reducible to a $\mathbf{FC}$ dark ceer.

Proof. If a minimal dark was reducible to a $\mathbf{FC}$ dark ceer, then some non-computable class would be reduced to a finite (computable) one, which is impossible. □

Corollary 3.6. There are no minimal $\mathbf{FC}$ dark ceers.

Proof. Let E be a $\mathbf{FC}$ dark ceer and a_0, a_1 be representatives from distinct E-classes. Define a ceer F as E, where a_0 and a_1 are collapsed. Now define a computable function f as follows:

$$f(x) = \begin{cases} x, & \text{if } x \notin [a_0]_F, \\ a_0, & \text{if } x \in [a_0]_F. \end{cases}$$

It is easy to check that f is a reduction from F to E. The reduction from E to F does not exist because of self-fullness of E (recall that every dark ceer is self-full). □

We now proceed to the main result of the paper, namely, that there exist weakly precomplete minimal dark ceers.

Theorem 3.7. Let R be any non-universal ceer. Then there exist infinitely many non-weakly precomplete pairwise incomparable minimal dark ceers $(E_l)_{l\in\omega}$ such that $E_l \nleq R$ for every $l \in \omega$.

Proof. Let R be a non-universal ceer. To construct ceers E_l that are non-weakly precomplete, we define a computable diagonal function d for every E_l as follows:

$$d(x) = \begin{cases} x+1, & \text{if } x \text{ is even}, \\ x-1, & \text{otherwise}. \end{cases}$$

In the construction we will place a restraint for E_l-collapsing any pair $(2x, 2x+1)$ and this restraint will never be cancelled. This will ensure that E_l is non-weakly precomplete (see Proposition 1.1).

Now, we will construct ceers E_l ($l \in \omega$) satisfying the following requirements:

$D_{i,j}^l$: If W_i intersects infinitely many E_l-classes, then it intersects $[j]_{E_l}$.

$Q_e^{k,l}$: If $k \neq l$ then φ_e is not a reduction from E_k to E_l.

P_e^l: φ_e is not a reduction from E_l to R.

The Q-requirements guarantee that the ceers E_l are pairwise incomparable. The D-requirements ensure that the ceers E_l are minimal and dark. Fulfilling the P-requirements ensures that the ceers E_l remain outside of the lower cone of R.

Strategy for $D_{i,j}^l$.

1. Choose a large number $m > j+1$.
2. Check whether W_i and $[j]_{E_l}$ intersect. If they do, the requirement is satisfied. If not, proceed to Step 3.
3. Wait for some $x \in W_i$ such that $(x, a) \notin E_l$ for every $a \leq m$.
4. E_l-collapse classes $[x]_{E_l}$ and $[j]_{E_l}$.

Strategy for $Q_e^{k,l}$.

1. Choose new witnesses $2x_0$ and $2x_1$ and place a restriction $R(2x_0, 2x_1, l)$.
2. Wait for $\varphi(2x_i)$ to converge for all $i = 0,1$.
3. Assume $\varphi_e(2x_i) = y_i$ for every $i \leq 1$.

(a) If $(y_0, y_1) \notin E_k$, then E_l-collapse $2x_0$ and $2x_1$, and place a restraint $R(y_0, y_1, k)$.

(b) If $y_0\ E_k\ y_1$, then keep the restraint $R(2x_0, 2x_1, l)$.

Strategy for P_e^l.

1. Set $m := 1$.
2. Choose new witnesses $2x_0$ and $2x_1$ and place a restraint $R(2x_0, 2x_1, l)$.
3. Wait for $\varphi_e(2x_i)$ to converge for all $i \leq m$.
4. Assume $\varphi_e(2x_i) = y_i$ for all $i \leq m$.

(a) If for all $i, j \leq m$

$2x_i\ E_l\ 2x_j \Leftrightarrow y_i\ R\ y_j$

then we E_l-collapse every pair $(2x_i, 2x_j) \Leftrightarrow i\ U\ j$ (where U is a univer-sal ceer), we set $m := m + 1$, choose a new witness $2x_m$ and go to Step 3.

(b) Otherwise, the requirement is satisfied.

Construction

The construction proceeds in stages. More precisely, we will construct computable approximations $\{E_{l,s}\}_{l\in\omega}$ of E_l for all $l \in \omega$. We say that a requirement $R \in \{P_e^l, Q^{k,l}: e, k, l \in \omega\}$ is initialized at stage s meaning that it chooses a new set of parameters, and when R is initialized, the restraint imposed by R is canceled. Note that Q and P requirements must deal with restraints of the form $R(a, b, l)$ posed by higher priority strategies.

Stage 0. Initialize all R-requirements and define $E_{l,0} = Id$ for each l.

Stage $s + 1$. If $D_{i,j}^l$ acts for some i, j, l then perform $D_{i,j}^l$-action as described. Otherwise, if R is initialized choose new parameters for R as described in the strategy for R; if R is not initialized then take R-action.

End of stage. Define $E_{l,s+1} := E_{l,s} \cup \{\text{all pairs which have been } E_l\text{-collapsed at stage } s + 1\}$. Go to the next stage.

Verification

We show that each requirement is satisfied by the respective strategy and that they don't collapse numbers $2k$ and $2k + 1$ for every $k \in \omega$.

Lemma 3.8. If W_i intersects infinitely many E_l-classes, then it intersects $[j]_{E_l}$. More-over, the strategy $D_{i,j}^l$ doesn't collapse elements $2k$ and $2k + 1$ for every $k \in \omega$.

Proof. Towards a contradiction, assume that W_i intersects infinitely many E_l-classes and does not intersect the class $[j]_{E_l}$. This means that each time we check the intersection at Step 2 we get a negative answer, and we proceed to Step 3. At Step 3, we wait for some $x \in W_i$ such that $(x, a) \notin E_l$ for every $a \leq m$, to appear. Otherwise, we may collapse an element from W_i and j. Since we wait at Step 3 infinitely often, W_i can only intersect classes smaller than m, leading to a contradiction.

Therefore, we collapse classes $[x]_{E_l}$ and $[j]_{E_l}$ according to Step 4 of the strategy. Now, we show that classes $[x]_{E_l}$ and $[j]_{E_l}$ cannot be of the form $[2k]_{E_l}$ and $[2k + 1]_{E_l}$. Assume that $j = 2k$. Recall Step 3 which says $x \notin [a]_{E_l}$ for all $a \leq m$ where we chose m to be a number greater than $j + 1 = 2k + 1$. Thus, the number x cannot be in $[2k + 1]_{E_l}$.

Lemma 3.9. If φ_e is total, then φ_e is not a reduction from E_l to E_k. Moreover, the strategy does not collapse numbers $2k$ and $2k + 1$ for every $k \in \omega$.

Proof. Suppose φ_e is total. When $\varphi_e(a)$ and $\varphi_e(b)$ converge, either we do nothing, while keeping the restraint $R(a, b, l)$ if $\varphi_e(a)\ E_k\ \varphi_e(b)$ already holds, or we act if still $(\varphi_e(a), \varphi_e(b)) \notin E_k$. Either way, the Q-requirement succeeds in diagonalization which is then preserved by the restraints posed by the strategy.

The strategy never collapses pairs $(2x, 2x + 1)$ for any $k \in \omega$. The only way some of the elements can be collapsed by the strategy is at Step 3(a), and those elements are even.

Lemma 3.10. If φ_e is a reduction from E_l to R, then there is a reduction from U to R. Moreover, the strategy doesn't collapse the numbers $2k$ and $2k + 1$ for every $k \in \omega$.

Proof. Suppose φ_e is a reduction from E_l to R. Then for every $i \in \omega$ a parameter $2x_i$ is assigned. For every $i, j \in \omega$ we have $i\ U\ j \Leftrightarrow \varphi_e(2x_i)\ R\ \varphi_e(2x_j)$. Note that this way we can reduce U to R, contradicting to the fact that R is non-universal.

The strategy never collapses numbers $2k$ and $2k+1$. The only instance when some elements might get collapsed by the strategy occurs at Step 4(a), and in such case the collapsed elements are even. □

The proof of Theorem 3.8 is complete. □

Conclusion

This paper gives a positive answer to the question about existence of minimal dark equivalences that are not weakly precomplete. Moreover, we proved the existence of infinitely many incomparable non-weakly precomplete minimal dark ceers that avoid lower cone of a given non-universal ceer. We also showed that over any dark $\mathbf{FC}$ ceer, there exists an infinite antichain of dark $\mathbf{FC}$ ceers.

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МИНИМАЛ ҚАРАҢҒЫ ЕСЕПТЕЛІМДІ САНАЛЫМДЫ ЭКВИВАЛЕНТ ҚАТЫНАСТАРДЫҢ ҚҰРЫЛЫМЫ ЖАЙЫНДА

Аңдатпа

Есептелімді көшіруге қатысты есептелімді саналымды эквиваленттік қатынастардың (қысқаша ceers) құрылымы соңғы 25 жылда белсенді түрде зерттелуде. Эндрюс пен Сорбидің шолу еңбектері ceers құрылымына тән көптеген құрылымдық қасиеттерді ашып көрсетті. Олар супремум мен инфимумның бар-жоғын зерттей отырып, құрылымды анықталымды екі бөлікке бөлді: қараңғы (яғни, тиімді трансверсалі жоқ) және жарық (тиімді трансверсалі бар) эквиваленттіліктер. Сонымен қатар, Эндрюс пен Сорби минималды қараңғы есептелімді саналымды эквиваленттік қатынастардың (бұдан әрі – минималды қараңғы ceers) шексіз саны бар екенін дәлелдеді. Мұндағы минималдылық – бұл қатынастардың төменгі конусында тек кластар саны шектеулі болатын эквиваленттіктер ғана жатуы мүмкін деген мағынада. Мұндай минималды қараңғы эквиваленттіктердің маңызды қасиеті – кластардың кез келген жұбы есептелімді түрде ажыратылмайды. Сондай-ақ, ceers теориясында әлсіз жартылай толық эквиваленттік қатынастар да зерттелуде. Бұл есептелімді диагонал функциясы жоқ эквиваленттіктер. Аталған қатынастарда да кластардың кез келген жұбы есептелімді түрде ажыратылмайды. Осыған байланысты әлсіз жартылай толық, минималды қараңғы эквиваленттік қатынастардың бар-жоғы туралы сұрақ туындайды. Бұл мақалада аталған сұраққа оң жауап беріліп отыр. FC арқылы барлық класы ақырлы болатын есептелімді саналымды эквиваленттік қатынас белгіленсін. Эндрюс, Швебер және Сорби қараңғы FC эквиваленттіктерінің бар екенін көрсеткен болатын. Ал бұл зерттеуде біз кез келген қараңғы FC эквиваленттік қатынасының үстінде қараңғы FC эквиваленттік қатынастардан тұратын шексіз антитізбек (антишынжыр) бар екенін дәлелдейміз.

Тірек сөздер: эквиваленттік қатынас, есептелімді саналымды эквиваленттік қатынас, есептелімді көшіру, әлсіз жартытолық эквиваленттік қатынас.

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О СТРУКТУРЕ МИНИМАЛЬНЫХ ТЕМНЫХ ВЫЧИСЛИМО ПЕРЕЧИСЛИМЫХ ОТНОШЕНИЙ ЭКВИВАЛЕНТНОСТИ

Аннотация

Структура вычислимо перечислимых отношений эквивалентности относительно вычислимой сводимости (коротко – ceers) активно развивается на протяжении последних 25 лет. В обзоре Эндрюса и Сорби было показано множество структурных свойств структуры ceers. Эндрюс и Сорби исследовали существование супремумов и инфимумов. Они разделили структуру на две определимые части: темные (эквивалентности без эффективного трасверсаля) и светлые (с эффективным трансверсалем) эквивалентности и показали существование бесконечного числа минимальных (в том смысле, что строго под ними могут быть только конечные эквивалентности) темных ceers. Минимальные темные эквивалентности имеют следующее свойство: каждая пара классов вычислимо неотделима. Также в теории ceers изучаются слабо предполные эквивалентности (то есть те эквивалентности, для которых не существует вычислимых диагональных функций). Также у данных эквивалентностей каждая пара классов вычислимо неотделима. В связи с этим возникает вопрос о существовании минимальных темных эквивалентностей, не являющихся слабо предполными. В данной статье дается положительный ответ на этот вопрос. Через $\mathbf{FC}$ обозначим в.п. отношение эквивалентности все классы которого конечны. Эндрюс, Швебер, Сорби показали существование темных $\mathbf{FC}$ эквивалентностей. В этой статье доказывается, что над любой темной $\mathbf{FC}$ эквивалентностью существует бесконечная антицепь темных $\mathbf{FC}$ эквивалентностей.

Ключевые слова: отношение эквивалентности, вычислимо перечислимое отношение эквивалентности, вычислимая сводимость, слабо предполные отношения эквивалентности.

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