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GENERALIZED NUMBERING FOR LINEAR ORDERS

Abstract

We study spectre of Turing degrees permitting to construct numbeings for the set of all linear orders isomorphic to the standard order of natural numbers. It is known that the index set of all linear orders isomorphic to the standard order of natural numbers is Π_3-comlete. This mean that this set has no computable numberings. In this work we show that the set of all linear orders isomorphic to the standard order of naturals has $0''$-computable numbering, and has no $0'$-computable numberings. In the Bazhenov, Kalmurzayev and Torebekova's work they construct universal c.e. linear preorder in the structure under computably reducibility. They use the following fact: there is computable numbering for some subset S_0 of c.e. linear preorders such that any c.e. linear preorder lies in lower cone for some c.e. linear order from S_0. We show that the similar fact is not hold for the structure of all linear orders isomorphic to the standard order of naturals. Moreover, for this structure there is no $0'$-computable numbering with simiral fact.

Keywords: positive preorder, positive equivalence, positive linear preorder, computable reducibility.

Introduction

The work is devoted to the study of numberings for a natural subclass of positive binary relations – the class of recursive linear orders isomorphic to the least limit ordinal. Further, we will denote this class as Ω.

Recall that a set $R \subseteq \omega \times \omega$ is a linear preorder if R is a reflexive and transitive relation, and for any $x, y \in \omega$,

$$(x, y) \in R \vee (y, x) \in R.$$

Let R and S be binary relations on ω. We say that R is computable reducible to S (denoted as $R \leq_c S$) if there is a computable function $f(x)$ such that for all $x, y \in \omega$,

$$xRy \Leftrightarrow f(x)Sf(y).$$

The systematic research into c-degrees for positive equivalence relations began in [1, 2]. In recent years, papers [3, 4, 5, 6, 7, 8] have investigated c-degrees of positive preorders.

The work of [9] initiated the investigation into the existence of joins and meets of positive linear preorders with respect to $\leq_c$. Furthermore, in the final section of this work, these questions were explored within the structure $(\Omega, \leq_c)$. Subsequently, the works of [8] and [10] continued the study of this structure. In [10], they measured the exact algorithmic complexities of index sets for all self-full recursive linear orders in Ω.

It is well known that there is a computable numbering ν for the family of all positive preorders on ω. We denote by P_i the positive preorder with index i (i.e., we will consider $\nu(i) = P_i$). With this numbering, it is not hard to construct a universal positive preorder under $\leq_c$: the uniformly join of all P_i is a universal positive preorder. Unfortunately, there are no computable numberings for the family of all positive linear preorders. Nevertheless, in the paper [14], the authors construct a universal positive linear preorder. Specifically, they demonstrate that the modification of the uniformly join of all positive linear preorders L_i is universal, where L_i is a numbering for the subfamily of all positive linear preorders satisfying the condition:

for any positive linear preorder L there is i such that $L \leq_c L_i$. (1)

In paper [8], it is established that the structure $(\Omega, \leq_c)$ lacks a universal degree. Moreover, within this structure, no modification of uniformly joins exists. In this study, our focus lies on the spectrum of Turing degrees capable of computing a numbering ν for a certain subfamily of Ω, under conditions similar to (1):

for any recursive order $L \in \Omega$ there is i such that $L \leq \nu(i)$ (2)

Section 2 provides the necessary preliminaries. In addition, Section 2 also gives a brief overview of the known results on the structure Celps. In Section 3, we prove that the family of all positive linear preorders has no computable numberings.

Materials and Methods

If R is a positive preorder, then a computable approximation of the preorder R is a uniform sequence of recursive preorders $(R^s)_{s\in\omega}$ with the following properties for each $s \in \omega$:

(1) $R^0 = Id \text{ and } R^s \subseteq R^{s+1}$;

(2) any set $R^{s+1} \setminus R^s$ is finite;

(3) $R = \bigcup_{s\in\omega} R^s$.

As we said in the introduction, by P_i we denote positive preorder with the index i. For given class $K \subseteq \{P_x : x \in \omega\}$, we say that $I_K = \{x : P_x \in K\}$ is an index set of the class K.

Assume L and R are linear orders on sets A and B respectively and $A \cap B = \emptyset$. Then $L + R$ is linear order on $A \cup B$ defined by the following:

$(x, y) \in L + R \Leftrightarrow [x \in A \& y \in B] \vee [x, y \in A \,\&\, (x, y) \in L] \vee [x, y \in B \,\&\, (x, y) \in R]$.

If L is a partial order on a set A and $s \in \omega$, then by $L \upharpoonright s$ we denote partial order on the set $A \cap \{x : x \leq s\}$ defined as L.

Let us recall some basic notions of the theory of numberings. Let A be a set of natural numbers and let S be an at most countable family of subsets of ω. A numbering $\nu : \omega \to S$ of a family S is called A-computable if the set $\{(k, x) : k \in \omega, x \in \nu(k)\}$ is A-computable enumerable, i.e. there is an uniformly effective procedure for numbering the sequence $\nu(0), \nu(1), \nu(2), \ldots$ of the sets from S. A family S is A-computable if it has at least one A-computable numbering. A numbering ν is said to be reducible to a numbering μ, briefly $\nu \leq \mu$, if there is a computable function $f(x)$ such that $\nu(k) = \mu(f(k))$ for all $k \in \omega$, and numberings ν, μ are said to be equivalent if $\nu \leq \mu$ and $\mu \leq \nu$. An A-computable numbering ν of a family S is universal if any A-computable numbering of S is reducible to ν.

In paper [1] it was proved that the family Ceers of all positive equivalence relations has a universal computable numbering. With a similar argument we can show that also the family Ceprs has a universal computable numbering. In [11] the numberings for the family Celps containing all positive linear preorders were investigated and the following fact was established.

Theorem 1 [11]. There is no computable numbering of the family of all positive linear preorders.

Naturally, the following question arises: what are oracles A such that the family Celps has an A-computable numbering?

Recall that the following result was proven in [13].

Theorem 3 [13]. For an arbitrary $A \subseteq \omega$, the following conditions are equivalent:

(1) $0'' \leq_T A'$;

(2) the family of all total recursive unary functions has an A-computable numbering;

(3) the family of all $\{0,1\}$-valued total recursive unary functions has an A-computable numbering.

Using Theorem 3, in paper [12], was established a criterion for the existence of a generalized computable number for the family of all positive linear preorders.

Theorem 2 [12]. Let A be an arbitrary oracle. The family of all positive linear preorders L has an A-computable numbering if and only if $0'' \leq_T A'$.

The article focuses on a specific order L: we consider the following substructure in the poset Celps.

Definition. By Ω we denote the following poset:

$$\Omega = (\{\deg_c(\mathrm{L}): \mathrm{L} \text{ is a computable linear order isomorphic to } \omega_{st}\}; \leq_c),$$

where ω_{st} standard order of natural numbers.

Based on the previous discussion, also the following question arises: for which oracles A does there exist an A-computable numbering of the family Ω? We will get the answer to this question in Theorem 4.

Result and Discussion

Theorem 4. Let A be an arbitrary set. The family of all positive linear preorders Ω has A-computable numbering if $A \geq_T 0''$.

Proof: There is a computable numbering of all c.e. preorders. Obviously, $\Omega \subseteq \{P_i : i \in \omega\}$. The condition $P_i \in \Omega$ is equivalent to the following: P_i is antisymmetric & P_i is linear & there is the least element with respect to P_i & any interval with respect to P_i is finite. More formally,

$$P_i \in \Omega \Leftrightarrow \forall x, y[xP_i y \,\&\, yP_i x \to x = y] \;\&$$

$$\& \;\forall x, y[xP_i y \vee yP_i x] \;\&\; \exists m \forall x[mP_i x] \;\& \forall x \exists y \forall z[z > y \to xP_i z]$$

Note that if P_i is antisymmetric and linear relation, then P_i is recursive.

Now for a given $i \in \omega$ we construct the numbering $\nu(i)$ as follows: we define

$$\nu_{s+1}(i) = \begin{cases} P_i \restriction s, & if\ \exists y \forall z[z > y \to\ sP_i z]\ true \\ \nu_s(i) + \{s+1\}, & if\ \exists y \forall z[z > y \to\ sP_i z]\ false \end{cases}$$

if

$$\forall x, y[xP_i y \,\&\, yP_i x \to x = y] \;\&\; \forall x, y[xP_i y \vee yP_i x] \;\&\; \exists m \forall x[mP_i x] \qquad (3)$$

is hold and otherwise we consider $\nu(i) = \omega_{st}$. Here ω_{st} is the standard order on the natural numbers. Define $\nu(i) = \cup_{s \in \omega} \nu_s(i)$.

The condition (3) is $0''$-computable, because P_i is recursive relation when P_i is antisymmetric and linear. So, it is not hard to see that the sequence $\{\nu_i\}_{i \in \omega}$ is uniformly $0''$-computable.

Assume $L \in \Omega$ is arbitrary linear order. It is clear that $L = P_{i_0}$ for some $i_o \in \omega$. For i_o the condition (3) is hold and moreover the condition $\exists y \forall z[z > y \to sP_{i_0} z]$ is also hold for any $s \in \omega$. This mean $\nu(i_0) = P_{i_0}$. So, $\nu \in Com^A(\Omega)$. ∎

Further, we will discuss questions for existence subfamilies of Ω with the condition (2).

Theorem 5. If $A \leq_T 0'$ then there is no subfamily S_0 of Ω and A-computable numbering ν for family S_0 such that

$$\text{for any recursive order } L \in \Omega \text{ there is } i \text{ such that } L \leq \nu(i). \quad (4)$$

Proof: By contradiction, assume $A = 0'$ and there is subfamily $S_0 \subset \Omega$ and $0'$-computable numbering ν for family S_0 satisfying (4).

Then no hard to show that for any i

$P_i \in \Omega \leftrightarrow P_i$-linear order and $P_i \leq_c \nu(n)$ for some n.

The last is equivalent to

$$(\forall x, y)[xP_iy \vee yP_ix] \;\&\; P_i - antisymmetric \;\&\; (\exists f - total)\ (\forall x, y)[xP_iy \leftrightarrow f(x)\nu(n)f(y)]$$

Here,

$(\forall x, y)[xP_iy \vee yP_ix]$ is Π_2 sentence;

$P_i - antisymmetric$ is Π_2 sentence;

$(\exists f) - total\ (\forall x, y)[xP_iy \leftrightarrow f(x)\nu(n)f(y)]$ is Σ_3 sentence; because $xP_iy \leftrightarrow f(x)\nu(n)f(y)$ is $0'$-computable. It follows that, $I_\Omega \in \Sigma_3^0$. Recall that in the work [10] authors showed that I_Ω is Π_3^0-complete set. Contradiction.

Note the following important corollaries.

Corollary 1. There is no A-computable numbering for the family Ω if $A \leq_T 0'$.

Corollary 2. There is no A-computable numbering ν for any subfamily Ω satisfying the condition (4).

Conclusion. We show that the family Ω has A-computable numbering if $A \geq_T 0''$. If $A \geq_T 0''$ then the condition (2) satisfied for any A-computable numbering of Ω. Also, we show that if $A \leq_T 0'$, then the family Ω has no A-computable subfamily satisfying condition (2).

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ОБОБЩЕННАЯ НУМЕРАЦИЯ ДЛЯ ЛИНЕЙНЫХ ПОРЯДКОВ

Аннотация

Мы изучаем спектр степеней Тьюринга, позволяющих построить нумерацию для множества всех линейных порядков, изоморфных стандартному порядку натуральных чисел. Известно, что индексное множество всех линейных порядков, изоморфных стандартному порядку натуральных чисел, является Π_3-полным. Это означает, что данное множество не имеет вычислимых нумераций. В данной работе мы показываем, что множество всех линейных порядков, изоморфных стандартному порядку натуральных чисел, имеет $0''$-вычислимую нумерацию и не имеет $0'$-вычислимых нумераций. В работах Баженова, Калмурзаева, Торебековой конструируются универсальные в.п. линейный предпорядок в структуре при вычислимой сводимости. Они используют следующий факт: существует вычислимая нумерация некоторого подмножества S_0 в.п. линейных предпорядков такая, что любой в.п. линейный предпорядок лежит в нижнем конусе для некоторого в.п. линейный порядок от S_0. Мы показываем, что аналогичный факт не имеет места для структуры всех линейных порядков, изоморфных стандартному порядку натуральных чисел. Более того, для этой структуры не существует $0'$-вычислимой нумерации с подобным фактом.

Ключевые слова: позитивный предпорядок, позитивная эквивалентность, позитивный линейный предпорядок, вычислимая сводимость.

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СЫЗЫҚТЫҚ РЕТТЕР ҮШІН ЖАЛПЫ НӨМІРЛЕУЛЕР

Аңдатпа

Біз натурал сандардың стандартты ретіне изоморфты барлық сызықтық реттердің жиыны үшін нөмірлеу құруға мүмкіндік беретін Тьюринг дәрежесінң спектрін зерттейміз. Натурал сандардың стандартты ретіне изоморфты барлық сызықтық реттердің индекстер жиыны Π_3-толық екені белгілі. Бұл жиында есептелімді нөмірлер жоқ дегенді білдіреді. Бұл жұмыста біз натуралдардың стандартты ретіне изоморфты барлық сызықтық реттердің жиынында $0''$-есептелімді нөмірленуі бар екенін және $0'$- есептелімді нөмірленуі жоқ екенін көрсетеміз. Баженов, Калмурзаев, Төребекованың еңбектерінде есептелімді көшулер кезінде универсал е.с. сызықтық жарты реттер құрастырылған. Олар келесі фактіні пайдаланады: е.с. кейбір S_0 ішкі жиыны үшін есептелетімді нөмірлеу табылады, сызықтық жарты реттер кейбір е.с. кейбір сызықтық реттер конусында төмен орналасқан . Біз натуралдардың стандартты ретіне изоморфты барлық сызықтық реттердің құрылымы үшін ұқсас факт орындалмайтынын көрсетеміз. Сонымен қатар, бұл құрылым үшін ұқсас фактісі бар $0'$-есептелетін нөмірлеу жоқ.

Тірек сөздер: позитивті жарты реттер, позитивті эквиваленттер, позитивті сызықтық жарты реттер, есептелімді көшулер.

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