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STRONGLY MINIMAL PARTIAL ORDERINGS OF HEIGHT TWO

Abstract

In the present paper, we study strongly minimal partial orderings in the signature containing only the symbol of binary relation of partial order. We use for partial orderings such characteristics as the height of a structure that is the supremum of lengths of ordered chains, and the width of a structure that is the supremum of lengths of antichains, where an antichain is a set of pairwise incomparable elements. We also differ trivial width and non-trivial width. Recently, B.Sh. Kulpeshov, In.I. Pavlyuk and S.V. Sudoplatov described strongly minimal partial orderings having a finite non-trivial width. Here we study strongly minimal partial orderings having an infinite non-trivial width. The main result of the paper is a criterion for strong minimality of an infinite partial ordering of height two having an infinite non-trivial width.

Key words: strongly minimal structure, partial ordering, connected component, maximal element, minimal element.

Introduction

Recall [1] that an infinite structure M is said to be minimal if for any formula φ(x) of the language of M, with parameters from M, either φ(M) or ¬φ(M) is finite. A theory T without finite models is said to be strongly minimal if any model M of T is minimal. Models of a strongly minimal theory are said to be strongly minimal, too.

Recall that a partial order on a set is a binary relation < satisfying:

Asymmetry $\forall x \forall y [x < y \to \neg (y < x)]$;

Transitivity $\forall x \forall y \forall z [(x < y \land y < z) \to x < z)]$.

A set equipped with a partial order on it is a partially ordered set or partial ordering.

Let M = ⟨M, <⟩ be an infinite partial ordering. Assuming that M is strongly minimal we have only finite ≤-chains and the lengths of these chains are bounded, since unbounded lengths imply existence of structures N that are elementarily equivalent to M with infinite chains $\{a_i \mid i \in Z\}$, $a_i < a_j$ for $i < j$, that violate the strong minimality by any formula $x < a_i$. Thus, the height h(M), that is the supremum of lengths of ≤-chains in M, must be finite for a strongly minimal structure M. Therefore, further we consider here only infinite partial orderings with finite ≤-chains.

Also, since M is infinite, it has an infinite width w(M) that is the supremum of cardinalities of ≤-antichains.

For any infinite partial ordering with finite ≤-chains M = ⟨M, <⟩ we can consider the corresponding graph structure MR = ⟨M, R2⟩, where R(a, b) iff either a is an immediate predecessor of b or a is an immediate successor of b with regard to the partial order < for any a, b ∈ M. We say a subset A of a partial ordering M = ⟨M, <⟩ is a connected component if A is a connected component in $M_R = \langle M, R^2\rangle$.

We say a connected component is trivial if it is a chain. We say a trivial connected component is an n-component if it is a chain of length n, where $1 \le n < \omega$. We say a 1-component is a singleton.

The value w(M) is witnessed by both trivial connected components and collections of maximal antichains in non-trivial connected components. Indeed, let M consist of both $\cup A_i$ and $\cup B_j$, where each A_i is a trivial connected component, and each B_j is a non-trivial connected component. Then obviously

$$w(M) = w_0(M) + w_1(M),$$

where $w_0(M)$ is the width of trivial part $\cup A_i$ (trivial width) and $w_1(M)$ is the width of non-trivial part $\cup B_j$ (non-trivial width).

The following theorem describes strongly minimal partial orderings with finite non-trivial width:

Theorem 1. [2, 3] An infinite partial ordering $M = \langle M, < \rangle$ with a finite non-trivial width is strongly minimal iff M has infinitely many singletons and additionally can have only finitely many finite connected components which are not singletons.

Thus, any strongly minimal partial ordering M with a finite non-trivial width is represented as a disjoint union $A(M) \cup B(M)$ of the following two parts:

1) A(M) is a disjoint union of infinitely many singletons;
2) B(M) is a disjoint union of finitely many, possibly zero-many, finite connected components which are not singletons.

In the recent paper [3], strongly minimal partial orderings having a finite non-trivial width were completely described. Here we study strongly minimal partial orderings having an infinite non-trivial width. In the present time, the class of strongly minimal theories is an object of active investigations. In [4], rank properties for families of strongly minimal theories were studied. In [5], some interesting results on strongly minimal structures were obtained that lead to a finer classification of strongly minimal structures with flat geometry.

Materials and Methods

Here we study partial orderings in the signature containing only the binary relation of partial order. Methods using for studying are classical methods of Model Theory, in particular, methods of studying strongly minimal structures.

Results and Discussion

Recall that an element a of a partial ordering M is said to be minimal if there is no element in M that is less than a. Also, an element a of a partial ordering M is said to be maximal if there is no element in M that is greater than a. Observe that any singleton of a partial ordering is both maximal and minimal element of the ordering. Also, any maximal (minimal) element of a non-trivial connected component of a partial ordering is not minimal (maximal).

Example 2 Let $M = \langle M, < \rangle$ be a partial ordering consisting of exactly one infinite non-trivial connected component of height two having exactly one maximal element. Then, obviously, M is a strongly minimal structure having an infinite non-trivial width.

Example 3. Let $M = \langle M, < \rangle$ be an infinite partial ordering consisting of exactly one infinite non-trivial connected component of height two having both infinitely many maximal elements and infinitely many minimal elements.

Consider the following formulas:

$$y \lozenge x := y \neq x \wedge \neg (y < x) \wedge \neg (x < y),$$
$$\mathrm{Min}(x) := \forall y\, [y \neq x \rightarrow y > x \vee y \lozenge x],$$
$$\mathrm{Max}(x) := \forall y\, [y \neq x \rightarrow y < x \vee y \lozenge x].$$

Obviously, Min(x) defines all minimal elements, and Max(x) defines all maximal elements of the structure. Both Min(M) and Max(M) are infinite, and $Min(M) \cap Max(M) = \varnothing$, whence M is not strongly minimal.

Example 4. Let $M = \langle M, < \rangle$ be an infinite partial ordering consisting of two infinite non-trivial connected components B_1 and B_2 of height two, both components contain exactly one maximal element.

Let a be the maximal element of B_1.

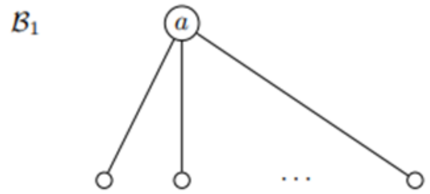

Figure 1 – Example 4

Consider the following formula: $\varphi(x, a) := x < a$.

Obviously, both $\varphi(M, a)$ and $\neg\, \varphi(M, a)$ are infinite, whence M is not strongly minimal.

We say that an element a of a partial ordering M has up-degree (down-degree) m for some $m \in \omega$ if there exist exactly m elements of M being an immediate successor (predecessor) of a. If there are infinitely many elements of M being an immediate successor (predecessor) of a then we say a has infinite up-degree (down-degree). Obviously, a has both up-degree 0 and down-degree 0 iff a is a singleton.

Example 5 Let $M = \langle M, < \rangle$ be an infinite partial ordering being a non-trivial connected component of height two and having exactly two maximal elements: both maximal elements have an infinite down-degree, and there exist both infinitely many minimal elements of up-degree 1 and infinitely many minimal elements of up-degree 2.

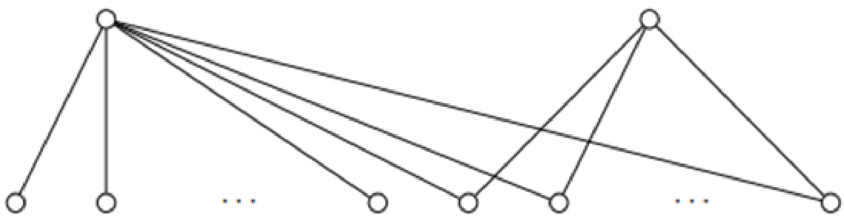

Figure 2 – Example 5

Consider the following formulas:

$$S(x, y) := x < y \wedge \forall\, t\, [x \le t \le y \rightarrow t = x \vee t = y],$$

$$D_1(x) := Min(x) \wedge \exists\, y\, [S(x, y) \wedge \forall\, t\, (S(x, t) \rightarrow t = y)],$$

$$D_2(x) := Min(x) \wedge \exists\, y_1\, \exists y_2\, [y_1 \ne y_2 \wedge S(x, y_1) \wedge S(x, y_2) \wedge \forall\, t\, (S(x, t) \rightarrow t = y_1 \vee t = y_2)].$$

Obviously, both $D_1(M)$ and $D_2(M)$ are infinite, and $D_1(M) \cap D_2(M) = \varnothing$, whence M is not strongly minimal.

Example 6. Let $M = \langle M, < \rangle$ be an infinite partial ordering being a non-trivial connected component of height two and having exactly two maximal elements: both maximal elements have an infinite down-degree; there exists a unique minimal element having up-degree 2, and the remaining minimal elements have up-degree 1. Let a be one of the maximal elements:

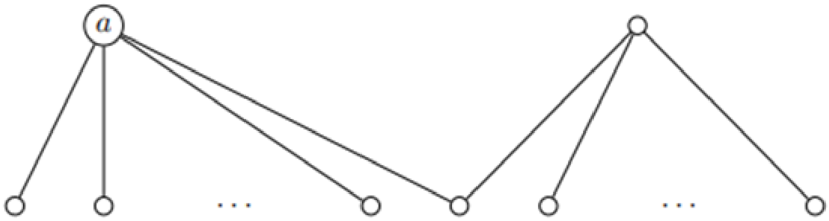

Figure 3 – Example 6

Consider the following formula: φ(x, y):= x < y.

Obviously, both φ(M, a) and ¬ φ(M, a) are infinite for any maximal element a of M, whence M is not strongly minimal.

Theorem 7. Let M = ⟨M, <⟩ be an infinite partial ordering of height two having an infinite non-trivial width. Then M is strongly minimal iff the following holds:

(1) M contains exactly one non-trivial connected component of height two having both an infinite set of maximal (minimal) elements and a non-empty finite set of minimal (maximal) elements;

(2) If M contains exactly m minimal (maximal) elements of infinite up-degree (down-degree) for some $1 \leq m < \omega$ then almost all maximal (minimal) elements have down-degree (up-degree) m;

(3) M contains only finitely many finite connected components of height at most two.

Proof of Theorem 7. (⇒) Let M be a strongly minimal structure. Since M has an infinite non-trivial width, we have that M contains infinitely many non-trivial connected components or at least one infinite non-trivial connected component. If M contains infinitely many non-trivial connected components then we obtain that M contains both infinitely many minimal elements and infinitely many maximal elements. This contradicts the strong minimality of M. Consequently, M contains only finitely many non-trivial connected components.

Further, since M has an infinite non-trivial width, at least one of non-trivial connected components is infinite. Denote it by B1. Since M has height two, B1 also has height two, and consequently B1 must contain either infinitely many maximal elements or infinitely many minimal elements. Suppose the first. Then Max(B1) is infinite, whence Max(M) also is infinite, where Max(x) is defined in Example 3. We assert that B1 contains only finitely many minimal elements. If this would be not true, then we would obtain that ¬Max(M) is infinite contradicting the strong minimality of M.

Prove now that M contains exactly one infinite non-trivial connected component of height two. Assume the contrary: let B1 and B2 be infinite non-trivial connected components of height two of the structure M. Suppose that B_1 contains infinitely many maximal elements. Then B1 contains finitely many minimal elements. Consequently, there exist a minimal element *b* having an infinite up-degree. Then we consider the following formula:

φ(x, b):= b < x.

Obviously, both φ(M,b) and ¬ φ(M, b) are infinite, whence M is not strongly minimal. Thus, M contains exactly one infinite non-trivial connected component of height two.

Let B be a unique infinite non-trivial connected component of height two, and suppose that B contains only finitely many minimal elements. Then we have that at least one of them has an infinite up-degree. If there exists exactly one minimal element having an infinite up-degree, the condition (2) holds. Suppose that there exist at least two minimal elements a, b ∈ B having an infinite up-degree. Consider the following formula:

φ(x, y):= y < x.

Obviously, both φ(B, a) and φ(B, b) are infinite. If at least one of the following sets φ(B, a) \ φ(B, b) and φ(B, b) \ φ(B, a) is infinite then we have that M is not strongly minimal. Therefore, both φ(B, a) \ φ(B, b) and φ(B, b) \ φ(B, a) are finite, i.e. almost all (except finitely many) maximal elements have the down-degree m.

If M contains infinitely many singletons then by using the fact that M contains at least one infinite non-trivial connected component we again obtain a contradiction with the strong minimality of M. We have similar reasons if M contains infinitely many 2-components. If M contains infinitely many finite non-trivial connected components then we obtain that M contains both infinitely many maximal elements and infinitely many minimal elements contradicting again the strong minimality of M. Thus, M can contain only finitely many finite connected components of height at most two.

(⇐) It can be proved that any infinite partial ordering M satisfying the conditions (1)-(3) is strongly minimal.

Recall that a first order theory is said to be totally categorical if it has exactly one model in each infinite power.

Corollary 7. Any strongly minimal partial ordering of height two having an infinite non-trivial width is totally categorical.

Conclusion

Thus, we obtained a complete description of strongly minimal partial orderings of height two having an infinite non-trivial width. As a corollary, we have that such strongly minimal partial orderings are totally categorical.

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СИЛЬНО МИНИМАЛЬНЫЕ ЧАСТИЧНЫЕ ПОРЯДКИ ВЫСОТЫ ДВА

Аннотация
В настоящей статье мы исследуем сильно минимальные частичные порядки в сигнатуре, содержащей только символ бинарного отношения, выражающего частичный порядок. Мы используем для частичных

порядков такие характеристики, как высота структуры, означающая супремум длин упорядоченных цепей, и ширина структуры, означающая супремум длин антицепей, где антицепь – это множество попарно несравнимых элементов. Мы также различаем как тривиальную ширину, так и нетривиальную ширину. Недавно Кулпешов Б.Ш, Павлюк Ин.И. и Судоплатов С.В. описали сильно минимальные частичные порядки, имеющие конечную нетривиальную ширину. Здесь мы исследуем сильно минимальные частичные порядки, имеющие бесконечную нетривиальную ширину. Основной результат статьи – критерий сильной минимальности бесконечного частичного порядка высоты два, имеющие бесконечную нетривиальную ширину.

Ключевые слова: сильно минимальная структура, частичный порядок, связная компонента, максимальный элемент, минимальный элемент.

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БИІКТІЛІГІ ЕКІ КҮШТІ МИНИМАЛДЫ ЖАРТЫЛАЙ РЕТТІЛІКТЕР

Аңдатпа

Бұл зерттеуде біз жартылай ретті білдіретін бинарлық қатынастың таңбасымен ғана анықталатын қолтаңбалардағы күшті минималды жартылай реттіліктерді талдаймыз. Жартылай реттіліктер үшін құрылымның биіктігі (реттелген тізбектердің ұзындықтарының жоғарғы шегі) және құрылымның ені (антитізбектердің ұзындықтарының жоғарғы шегі) сияқты сипаттамалар қолданылады. Мұнда антитізбек деп өзара салыстырылмайтын элементтер жиынтығы түсіндіріледі. Зерттеу барысында біз тривиалды және тривиалды емес енді жартылай реттіліктерді ажыратамыз. Бұған дейін Б.Ш. Кулпешов, Ин.И. Павлюк және С.В. Судоплатов шекті тривиалды емес ені бар күшті минималды жартылай реттіліктерді сипаттаған болатын. Осы жұмыста біз шексіз тривиалды емес ені бар күшті минималды жартылай реттіліктерді қарастырамыз. Мақаланың негізгі нәтижесі – шексіз тривиалды емес ені бар және биіктігі екіге тең шексіз жартылай реттіліктердің күшті минималдылығы үшін қажетті және жеткілікті шарттың анықталуы.

Тірек сөздер: күшті минималды құрылым, жартылай реттілік, жалғанған компонент, максималды элемент, минималды элемент.

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