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STABILITY CONDITION OF FINITE DIFFERENCE SCHEMES FOR PARABOLIC AND HYPERBOLIC EQUATIONS: A COMPARISON WITH FINITE VOLUME METHODS FOR FRACTIONAL-ORDER DIFFUSION

Abstract

This paper compares the finite difference and finite volume methods for solving time-fractional diffusion equations. These methods are widely known for diffusion equations with integer order, but their effectiveness for time-fractional diffusion equations has not been sufficiently studied. The definition of the Grunwald-Letnikov fractional derivative is used to approximate the equation. An explicit difference scheme for the finite difference method is obtained and a stability condition for the fractional time order difference scheme is derived, which is also a generalisation for parabolic and hyperbolic type equations, which was previously unknown for schemes with a fractional time order. An explicit discrete form for solving subdiffusion equations in two-dimensional space with fractional time order by the finite volume method is presented. Numerical results show that the finite difference method demonstrates high accuracy, while the finite volume method is better suited for complex geometries. These findings provide insights for future developments in anomalous diffusion modeling.

Key words: subdiffusion, finite difference method, finite volume method, Grunwald-Letnikov fractional derivative, stability condition.

Introduction

Fractional derivatives are applied from fundamental laws of natural science including physics, biology, chemistry up to equations of economics and finance. For example, the equation for the electric field strength in one-dimensional space proposed by Westerlund S. [1] has the form

$$\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} + \mu_0 \varepsilon_0 \epsilon_0 E^\alpha + \frac{\partial^2 E}{\partial x^2} = 0, \quad (1)$$

where E is the electric field strength, μ_0, ε_0 and ϵ_0 are constants, E^α is the fractional derivative of the electric field strength of order α $(0 < \alpha < 2)$.

Also, in his studies [2] we notice that he replaces fractional derivatives in Maxwell's equations, specifically the relations $D = \varepsilon E$, $B = \mu H$(where D is the electric displacement and B is the magnetic induction), with $D = \varepsilon E^{(\alpha-1)}$ and $B = \mu H^{(\alpha-1)}$. This idea was further developed in [3]. Westerlund S. [2] considered that the theory of elasticity ($F = kx$), the Newtonian model of viscous fluid ($F = kx'$), and Newton's second law ($F = kx''$) lead to the conclusion that it is possible to generalize Hooke's law in the form $F = kx^{\beta}$, where the order of the derivative β can be any real number. The importance of fractional calculus in cosmology was described in [4, 5]. Based on [6–16] we can see the importance of fractional derivatives in mechanics, for example, Bassett's problem on the motion of a sphere immersed in an incompressible viscous fluid [6]. In dimensionless form, it reduces to the differential equation

$$\frac{\partial V(t)}{\partial t} + \beta^{\alpha} \cdot \frac{\partial^{\alpha} V(t)}{\partial t^{\alpha}} + V(t) = 1,\ \beta^{\alpha} > 0,\ 0 < \alpha < 1,\ V(0^{+}) = V_0 \tag{2}$$

Lundstrom, Richner [17] show the connection of non-integer order derivatives with neocortical neurons. Harjule and Bansal [18] show fractional order models of viscoelasticity of lung tissue with force:

$$\sigma(t) = M\,\epsilon(t) + ND_t^{\alpha}[\epsilon(t)], \tag{3}$$

where σ is the stress, ϵ is the involved strain, M is the spring stiffness, N is the viscoelastic parameter of the system.

In biology and demography, the well-known Malthus law describes exponential population growth. In this work [19], the Fractional Malthus equation has been considered:

$$\left(D_{t_0}^{\alpha} P\right)(t) = r \cdot P(t), \tag{4}$$

where r is the growth rate and $0 \le \alpha \le 1$.

Applications of fractional calculus in different spheres of science are described in detail in [20–24]. In this paper we consider anomalous diffusion in two-dimensional space:

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,y,t) = \frac{\partial^2}{\partial x^2} u(x,y,t) + \frac{\partial^2}{\partial y^2} u(x,y,t) \tag{5}$$

If $0 < \alpha < 1$ then it is called subdiffusion, but if $1 < \alpha < 2$, then this phenomenon is called superdiffusion.

Nowadays the numerical solution of fractional order diffusion equations in terms of Caputo's derivative is presented in many papers [25–31], but many of them are based on the finite difference method. [25] demonstrates the convergence of difference schemes for the one-dimensional equation. Alikhanov A. [26] developed a new difference analogue of the Caputo fractional derivative for $0 < \alpha < 1$, which contributed to progress in this field. In [27, 29], schemes for fractional equations with delay have been constructed. Reference [28] examines the finite element method for a nonlinear parabolic equation of fractional order. Liu et al. [30] proposed implicit difference and explicit difference techniques to find the numerical solution of space-time fractional advection dispersion problem. A fractional analog of Crank Nicholson method was constructed to deal with the one-dimensional space two-sided space fractional diffusion equation with functional delay in [31]. The method of finite volumes also represents a huge role in numerical methods of solution of partial derivative equations, since it allows to work with complex geometrical shapes, and the law of conservation of the quantity of interest such as mass, momentum and energy is fulfilled.

The purpose of this paper is to compare the efficiency of the finite difference method (FDM) and the finite volume method (FVM) in solving the fractional order time-dependent diffusion equations and to derive the stability condition for the explicit scheme of equation (5). Similar comparisons for integer derivatives were made in [32–34]. The effectiveness of the finite volume method (FVM) for advection-diffusion equations is demonstrated in [32], however, we will show that this is not the case for subdiffusion equations. In [35] the comparison of FDM and FVM is given only for the superdiffusion equation.

Preliminaries. There are several definitions of fractional derivative, the best known are the definitions of Riemann-Liouville, Caputo, Grunwald-Letnikov. In this paper we use the Grunwald-Letnikov definition of fractional derivative [36,37] of the form

$$\left(D_{x_0}^{\alpha} f\right)(x) = \lim_{N\to\infty} \left(\frac{x-x_0}{N}\right)^{-\alpha} \frac{1}{\Gamma(-\alpha)} \sum_{k=0}^{N-1} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} f\left(x - k\frac{x-x_0}{N}\right) \quad (6)$$

$$\left(D_{x_0}^{\alpha} f\right)(x) = \lim_{h\to 0} (h)^{-\alpha} \frac{1}{\Gamma(-\alpha)} \sum_{k=0}^{N-1} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} f(x-kh) =$$

$$= \lim_{h\to 0} (h)^{-\alpha} \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} \sum_{k=0}^{N-1} f(x-kh) \quad (7)$$

where $h = \frac{x-x_0}{N}$.

The form of writing (7) does not provide a possibility to perform calculations for integers α since $\Gamma(-\alpha)$ is not defined, and requires the calculation of large numbers, so it is necessary to take

$$\frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} = (-1)^k \cdot \binom{\alpha}{k} = \frac{(-\alpha)\cdot(-\alpha+1)\cdots(-\alpha+k-1)}{k!} \quad (8)$$

The proof of equality (8) is given in [38,39].

Substituting in (7) the obtained equality (8) we obtain

$$\left(D_{x_0}^{\alpha} f\right)(x) = \lim_{h\to 0} (h)^{-\alpha} \sum_{k=0}^{N-1} \binom{\alpha}{k} f(x-kh) \quad (9)$$

Problem Statement

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + F(x,y,t) \quad (10)$$

where $\alpha = 0.5$, $F(x,y,t) = 1000 \cdot \left(\frac{2t^{\frac{3}{2}}}{3\sqrt{\pi}} + \frac{2}{\sqrt{\pi}}\sqrt{t}\right) \cdot y \cdot sinx + 1000 \cdot \left(\frac{t^2}{2} + t\right) \cdot y \cdot sinx$.

Initial condition: $u(x,y,0) = 0$;

Boundary conditions: $u(0,y,t) = 0,\ u(1,y,t) = 1000 \cdot \left(\frac{t^2}{2} + t\right) \cdot y \cdot sin1,$

$$u(x,0,t) = 0,\ u(x,1,t) = 1000 \cdot \left(\frac{t^2}{2} + t\right) \cdot sinx \ ;$$

Analytical solution:

$$u(x,y,t) = 1000 \cdot \left(\frac{t^2}{2} + t\right) \cdot y \cdot sinx \quad (11)$$

Materials and Methods

Solution of equation (10) by finite difference method.

The discrete analogue of the Grunwald-Letnikov fractional derivative definition (9):

$$\left(D_{x_0}^{\alpha} f\right)(x) \approx \frac{1}{\Delta h^{\alpha}} \sum_{k=0}^{n+1} u_{i,j}^{n+1-k} \binom{\alpha}{k} \quad (12)$$

An approximation of the second order derivative:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2} \quad (13)$$

Using (12) and (13) we find the finite-difference scheme for (10):

$$\frac{1}{\Delta t^{\alpha}}\sum_{k=0}^{n+1} u_{i,j}^{n+1-k}\binom{\alpha}{k} = \frac{u_{i-1,j}^{n}-2u_{i,j}^{n}+u_{i+1,j}^{n}}{\Delta x^2} + \frac{u_{i,j-1}^{n}-2u_{i,j}^{n}+u_{i,j+1}^{n}}{\Delta y^2} + F(x_i, y_j, t_n) \quad (14)$$

From (14) we can obtain the explicit scheme:

$$u_{i,j}^{n+1} =$$
$$= \Delta t^{\alpha}\cdot\left(\frac{u_{i-1,j}^{n}-2u_{i,j}^{n}+u_{i+1,j}^{n}}{\Delta x^2} + \frac{u_{i,j-1}^{n}-2u_{i,j}^{n}+u_{i,j+1}^{n}}{\Delta y^2} + F(x_i, y_j, t_n)\right) - \sum_{k=1}^{n+1} u_{i,j}^{n+1-k}\binom{\alpha}{k} \quad (15)$$

Stability analysis of the difference scheme (15):

The Grunwald-Letnikov approximation of the fractional derivative has a first order error. For details see Podlubny [40].

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{\Delta t^{\alpha}}\sum_{k=0}^{n+1} u_{i,j}^{n+1-k}\binom{\alpha}{k} + O(\Delta t) \quad (16)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1}-2u_i+u_{i+1}}{\Delta x^2} + O(\Delta x^2) \quad (17)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{j-1}-2u_j+u_{j+1}}{\Delta y^2} + O(\Delta y^2) \quad (18)$$

Let's have

$$N = D + \varepsilon, \quad (19)$$

where N is the solution obtained on the computer, D is the exact solution of the difference equation, ε is the rounding error.

Let us consider equation (14) in homogeneous form:

$$\frac{1}{\Delta t^{\alpha}}\sum_{k=0}^{n+1} u_{i,j}^{n+1-k}\binom{\alpha}{k} = \frac{u_{i-1,j}^{n}-2u_{i,j}^{n}+u_{i+1,j}^{n}}{\Delta x^2} + \frac{u_{i,j-1}^{n}-2u_{i,j}^{n}+u_{i,j+1}^{n}}{\Delta y^2} \quad (20)$$

Considering (19), (20) will take the form:

$$\frac{1}{\Delta t^{\alpha}}\sum_{=0}^{n+1} N_{i,j}^{n+1-k}\binom{\alpha}{k} + \varepsilon_{i,j}^{n+1-k}\binom{\alpha}{k} =$$
$$= \frac{N_{i-1,j}^{n}+\varepsilon_{i-1,j}^{n}-2N_{i,j}^{n}-2\varepsilon_{i,j}^{n}+N_{i+1,j}^{n}+\varepsilon_{i+1,j}^{n}}{\Delta x^2} + \frac{N_{i,j-1}^{n}+\varepsilon_{i,j-1}^{n}-2N_{i,j}^{n}-2\varepsilon_{i,j}^{n}+N_{i,j+1}^{n}+\varepsilon_{i,j+1}^{n}}{\Delta y^2} \quad (21)$$

$$\frac{1}{\Delta t^{\alpha}}\sum_{k=0}^{n+1} \varepsilon_{i,j}^{n+1-k}\binom{\alpha}{k} = \frac{\varepsilon_{i-1,j}^{n}-2\varepsilon_{i,j}^{n}+\varepsilon_{i+1,j}^{n}}{\Delta x^2} + \frac{\varepsilon_{i,j-1}^{n}-2\varepsilon_{i,j}^{n}+\varepsilon_{i,j+1}^{n}}{\Delta y^2} \quad (22)$$

Let's represent $\varepsilon(x, y, t)$ as a sum of Fourier series:

$$\varepsilon(x, y, t) = b_m(t)e^{il_m x}e^{il_m y} \quad (23)$$

Let's see. $\varepsilon_m(x, y, t) = \sum_m b_m(t)e^{il_m x}e^{il_m y}$. We will look for a solution in the form $z^n e^{il_m x}e^{il_m y}$. When $t = 0$ (n=0) it has the form $e^{il_m x}e^{il_m y}$. Let $z^n = e^{a\Delta t}$then

$$z^n = e^{an\Delta t} = e^{at},\ \varepsilon_m(x, y, t) = e^{at}e^{il_m x}e^{il_m y}, \quad (24)$$

l_m is real, a can be complex. Substituting (24) into (22), we obtain

$$\sum_{k=0}^{n+1} e^{a(t+\Delta t-k\Delta t)} e^{il_m x} e^{il_m y} \binom{\alpha}{k} =$$
$$= \Delta t^{\alpha} \left(\frac{e^{at} e^{il_m(x-\Delta x)} e^{il_m y} - 2e^{at} e^{il_m x} e^{il_m y} + e^{at} e^{il_m(x+\Delta x)} e^{il_m y}}{\Delta x^2} + \right.$$
$$\left. + \frac{e^{at} e^{il_m x} e^{il_m(y-\Delta y)} - 2e^{at} e^{il_m x} e^{il_m y} + e^{at} e^{il_m x} e^{il_m(y+\Delta y)}}{\Delta y^2} \right) \tag{25}$$

Dividing by $e^{at} e^{il_m x} e^{il_m y}$ and using the ratio

$$\cos\beta = \frac{e^{i\beta} + e^{-i\beta}}{2}$$

Get this:

$$\sum_{k=0}^{n+1} e^{a(\Delta t-k\Delta t)} \binom{\alpha}{k} = \Delta t^{\alpha} \left(\frac{2\cos l_m \Delta x - 2}{\Delta x^2} + \frac{2\cos l_m \Delta y - 2}{\Delta y^2} \right) \tag{26}$$

Using the trigonometric identity

$$sin^2\left(\frac{\beta}{2}\right) = \frac{1-\cos\beta}{2}$$

Let us rewrite the last relation in the following form:

$$e^{a\Delta t} + \sum_{k=2}^{n+1} e^{a(\Delta t-k\Delta t)} \binom{\alpha}{k} = \alpha - 4\frac{\Delta t^{\alpha}}{\Delta x^2} sin^2\left(\frac{l_m \Delta x}{2}\right) - 4\frac{\Delta t^{\alpha}}{\Delta y^2} sin^2\left(\frac{l_m \Delta y}{2}\right) \tag{27}$$

Let's represent the expression $\sum_{k=2}^{n+1} e^{a(\Delta t-k\Delta t)} \binom{\alpha}{k}$ as

$$\sum_{k=2}^{n+1} e^{a(\Delta t-k\Delta t)} \binom{\alpha}{k} = e^{a(-\Delta t)} \binom{\alpha}{2} + e^{a(-2\Delta t)} \binom{\alpha}{3} + e^{a(-3\Delta t)} \binom{\alpha}{4} + \cdots$$

Since for each harmonic $\varepsilon_i^{n+1} = e^{a\Delta t} \varepsilon_i^n$ then $|e^{a\Delta t}| \le 1$. Note that $|e^{a\Delta t(1-k)}| \ge 1$, $\binom{\alpha}{k} \le 0$ then $\left|\binom{\alpha}{k} e^{a\Delta t(1-k)}\right| \le \binom{\alpha}{k}$. Let us take each term of the left-hand side of equality (27) as a module.

Let's use the modulus property

$$|a_1 + a_2 + a_3 + \ldots + a_n| \le |a_1| + |a_2| + |a_3| + \cdots |a_n|$$

$$\left|e^{a\Delta t} + \sum_{k=2}^{n+1} e^{a(\Delta t-k\Delta t)} \binom{\alpha}{k}\right| \le |e^{a\Delta t}| + \left|\binom{\alpha}{3} e^{a(-2\Delta t)}\right| + \left|\binom{\alpha}{4} e^{a(-3\Delta t)}\right| + \left|\binom{\alpha}{5} e^{a(-4\Delta t)}\right| + \cdots$$

By defining $\sum_{k=2}^{n+1} \binom{\alpha}{k} \approx \alpha - 1$ we find out that the scheme is stable at

$$\left|\alpha - 4\frac{\Delta t^{\alpha}}{\Delta x^2} sin^2\left(\frac{l_m \Delta x}{2}\right) - 4\frac{\Delta t^{\alpha}}{\Delta y^2} sin^2\left(\frac{l_m \Delta y}{2}\right)\right| \le \alpha \tag{28}$$

Let's consider two possible cases:

1) $\alpha - 4\frac{\Delta t^{\alpha}}{\Delta x^2} sin^2\left(\frac{l_m \Delta x}{2}\right) - 4\frac{\Delta t^{\alpha}}{\Delta y^2} sin^2\left(\frac{l_m \Delta y}{2}\right) \le \alpha$

$$-4\frac{\Delta t^{\alpha}}{\Delta x^2} sin^2\left(\frac{l_m \Delta x}{2}\right) - 4\frac{\Delta t^{\alpha}}{\Delta y^2} sin^2\left(\frac{l_m \Delta y}{2}\right) \le 0$$

$$\frac{\Delta t^{\alpha}}{\Delta x^2} + \frac{\Delta t^{\alpha}}{\Delta y^2} \ge 0$$

2) $\alpha - 4\frac{\Delta t^\alpha}{\Delta x^2} sin^2\left(\frac{l_m \Delta x}{2}\right) - 4\frac{\Delta t^\alpha}{\Delta y^2} sin^2\left(\frac{l_m \Delta y}{2}\right) \leq -\alpha$

$-4\frac{\Delta t^\alpha}{\Delta x^2} sin^2\left(\frac{l_m \Delta x}{2}\right) - 4\frac{\Delta t^\alpha}{\Delta y^2} sin^2\left(\frac{l_m \Delta y}{2}\right) \leq -2\alpha$

$\frac{\Delta t^\alpha}{\Delta x^2} + \frac{\Delta t^\alpha}{\Delta y^2} \leq \frac{\alpha}{2}$

In the first case $\frac{\Delta t^\alpha}{\Delta x^2} + \frac{\Delta t^\alpha}{\Delta y^2} \geq 0$, and in the second case at $\frac{\Delta t^\alpha}{\Delta x^2} + \frac{\Delta t^\alpha}{\Delta y^2} \leq \frac{\alpha}{2}$. The last inequality is the stability condition of this scheme.

Solution of equation (10) by the finite volume method.

Let us introduce the Nabla operator for two-dimensional space:

$$\nabla u = \frac{\partial u}{\partial x}\bar{i} + \frac{\partial u}{\partial y}\bar{j} \qquad (29)$$

Equation (10) is transformed in integral form

$$\int_{V_0} \frac{\partial^\alpha u}{\partial t^\alpha} dV = \int_{V_0} \nabla \cdot (\nabla u) dV + \int_{V_0} F(x,y,t) dV \qquad (30)$$

Following the Gauss-Ostrogradsky theorem (30) will take the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} \cdot V_0 = \int_S \nabla u \cdot \bar{n} dA + F(x,y,t) \cdot V_0 \qquad (31)$$

Since the control volumes are bounded by a certain number of faces, the surface integral can be replaced by a discrete sum of sides.

$$\int_S \nabla u \cdot \bar{n} dA \approx \sum_f (\nabla u)_f \cdot \bar{n}_f \cdot A_f \qquad (32)$$

Correspondingly $\frac{\partial^\alpha u}{\partial t^\alpha} \cdot V_0 = \sum_f (\nabla u)_f \cdot \bar{n}_f \cdot A_f + F(x,y,t) \cdot V_0$ (33)

Since u is a scalar quantity, the gradient u is a vector, such a vector in the Cartesian coordinate system (x,y,z) can be written as

$$\nabla u = \frac{\partial u}{\partial x}\bar{i} + \frac{\partial u}{\partial y}\bar{j} + \frac{\partial u}{\partial z}\bar{k}, \qquad (34)$$

where $\bar{i}$, $\bar{j}$, $\bar{k}$ are unit pairwise orthogonal vectors.

∇u can be expressed in bases $\hat{n}$, $\hat{t}_1, \hat{t}_2$:

$$\nabla u = \frac{\partial u}{\partial x}\bar{n} + \frac{\partial u}{\partial y}\bar{t}_1 + \frac{\partial u}{\partial z}\bar{t}_2, \qquad (35)$$

where $\bar{n}$ is the unit normal vector, $\bar{t}_1, \bar{t}_2$ are unit tangents on the plane perpendicular to the plane of the normal vector.

It's worth noting $\frac{\partial u}{\partial x} = (\nabla u) \cdot \bar{i}$, $\frac{\partial u}{\partial y} = (\nabla u) \cdot \bar{j}$, $\frac{\partial u}{\partial z} = (\nabla u) \cdot \bar{k} \quad \Rightarrow$

$$\Rightarrow \quad \nabla u = [(\nabla u) \cdot \bar{i}]\bar{i} + [(\nabla u) \cdot \bar{j}]\bar{j} + [(\nabla u) \cdot \bar{k}]\bar{k} \qquad (36)$$

Similarly (35) can be converted to

$$\nabla u = [(\nabla u) \cdot \bar{n}]\bar{n} + [(\nabla u) \cdot \bar{t}_1]\bar{t}_1 + [(\nabla u) \cdot \bar{t}_2]\bar{t}_2 \qquad (37)$$

Let's consider ∇u in two-dimensional space, then

$$(\nabla u)_f = \left[(\nabla u)_f \cdot \bar{n}_f\right]\bar{n}_f + \left[(\nabla u)_f \cdot \bar{t}_f\right]\bar{t}_f \tag{38}$$

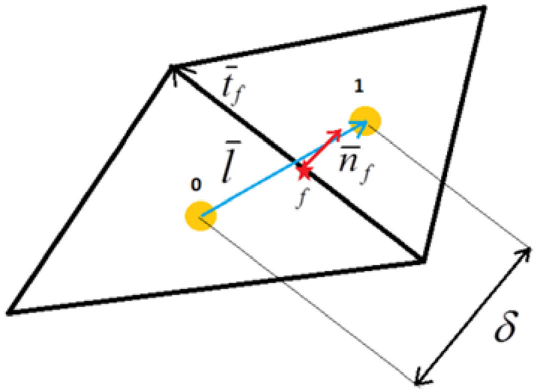

Figure 1 – Two neighboring cells of an unstructured grid

Consider a vector l which connects two centres of adjacent cells. Let's perform scalar multiplication l with equation (38)

$$(\nabla u)_f \cdot l = \left[(\nabla u)_f \cdot \bar{n}_f\right]\bar{n}_f \cdot l + \left[(\nabla u)_f \cdot \bar{t}_f\right]\bar{t}_f \cdot l \tag{39}$$

Note that the value $\bar{n}_f \cdot l = \delta$ since $\bar{n}_f \cdot l$ is a component of the vector l along the direction of normal to the surface, so it is the distance between the centre O and 1 measured in the direction of normal to the surface. Then

$$(\nabla u)_f \cdot l = \left[(\nabla u)_f \cdot \bar{n}_f\right]\delta + \left[(\nabla u)_f \cdot \bar{t}_f\right]\bar{t}_f \cdot l \tag{40}$$

To express $(\nabla u)_f \cdot l$ through the values of the centres of the cells, we decompose in Taylor series the centres O and 1 with respect to f. This procedure is similar to the definition of the central difference formulas for an orthogonal mesh.

$$u_1 = u_f + \frac{\partial u}{\partial x}\Big|_f (x_1 - x_f) + \frac{\partial u}{\partial y}\Big|_f (y_1 - y_f) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Big|_f (x_1 - x_f)^2 + + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}\Big|_f (y_1 - y_f)^2 + \frac{1}{2}\frac{\partial^2 u}{\partial x \partial y}\Big|_f (x_1 - x_f)(y_1 - y_f) + \cdots \tag{41}$$

$$u_O = u_f + \frac{\partial u}{\partial x}\Big|_f (x_O - x_f) + \frac{\partial u}{\partial y}\Big|_f (y_O - y_f) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Big|_f (x_O - x_f)^2 + + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}\Big|_f (y_O - y_f)^2 + \frac{1}{2}\frac{\partial^2 u}{\partial x \partial y}\Big|_f (x_O - x_f)(y_O - y_f) + \cdots \tag{42}$$

By subtracting equations (41), (42), we obtain

$$u_1 - u_O \approx \frac{\partial u}{\partial x}\Big|_f (x_1 - x_O) + \frac{\partial u}{\partial y}\Big|_f (y_1 - y_O) = (\nabla u)_f \cdot l \tag{43}$$

Based on equations (43) and (40)

$$(\nabla u)_f \cdot \bar{n}_f = \frac{u_1 - u_O}{\delta} - \frac{\left[(\nabla u)_f \cdot \bar{t}_f\right]\bar{t}_f \cdot l}{\delta} \tag{44}$$

From the obtained equality (44), where $(\nabla u)_f \cdot l = u_1 - u_O$ or $(\nabla u)_f \cdot |l|\bar{l} =$ $= u_1 - u_O \Rightarrow (\nabla u)_f \cdot \bar{l} = \frac{u_1 - u_O}{|l|}$.

Likewise $(\nabla u)_f \cdot \bar{t}_f = \frac{u_a - u_b}{|t_f|}$ (45)

Then equation (44) will take the form

$$(\nabla u)_f \cdot \bar{n}_f = \frac{u_1 - u_O}{\delta} - \left[\frac{u_a - u_b}{\delta |t_f|}\right] \bar{t}_f \cdot l_f \quad (46)$$

Substituting the obtained equation (46) into (32), we obtain

$$\frac{\partial^\alpha u}{\partial t^\alpha} \cdot V_0 = \sum_f \left(\frac{u_{nb(f)} - u_O}{\delta_f} - \left[\frac{u_{a(f)} - u_{b(f)}}{\delta_f |t_f|}\right] \bar{t}_f \cdot l_f\right) A_f + F(x, y, t) \cdot V_0 \quad (47)$$

Using (12) and (47) we find the explicit discrete form by the Jacobi method using an unstructured mesh:

$$\frac{1}{\Delta t^\alpha} \sum_{k=0}^{n+1} u_{i,j}^{n+1-k} \binom{\alpha}{k} \cdot V_i =$$

$$= \sum_f \left(\frac{u_{i,nb(f)}^n - u_i^n}{\delta_{i,f}} - \left[\frac{u_{i,a(f)}^n - u_{i,b(f)}^n}{\delta_{i,f} |\bar{t}_{i,f}|}\right] \cdot \bar{t}_{i,f} \cdot \bar{l}_{i,f}\right) \cdot A_{i,f} + F(x, y, t_n) \cdot V_i$$

or

$$u_i^{n+1} =$$

$$= \frac{\Delta t^\alpha}{V_i} \cdot \left(\sum_f \left(\frac{u_{i,nb(f)}^n - u_i^n}{\delta_{i,f}} - \left[\frac{u_{i,a(f)}^n - u_{i,b(f)}^n}{\delta_{i,f} |\bar{t}_{i,f}|}\right] \cdot \bar{t}_{i,f} \cdot \bar{l}_{i,f}\right) \cdot A_{i,f} + F(x, y, t_n) \cdot V_i\right) \quad (48)$$

$$- - \sum_{k=1}^{n+1} u_i^{n+1-k} \binom{\alpha}{k}$$

An analysis of the stability of the discrete form of FVM, only with the definition of Caputo's fractional derivative is given in [41].

Results and Discussion

FVM

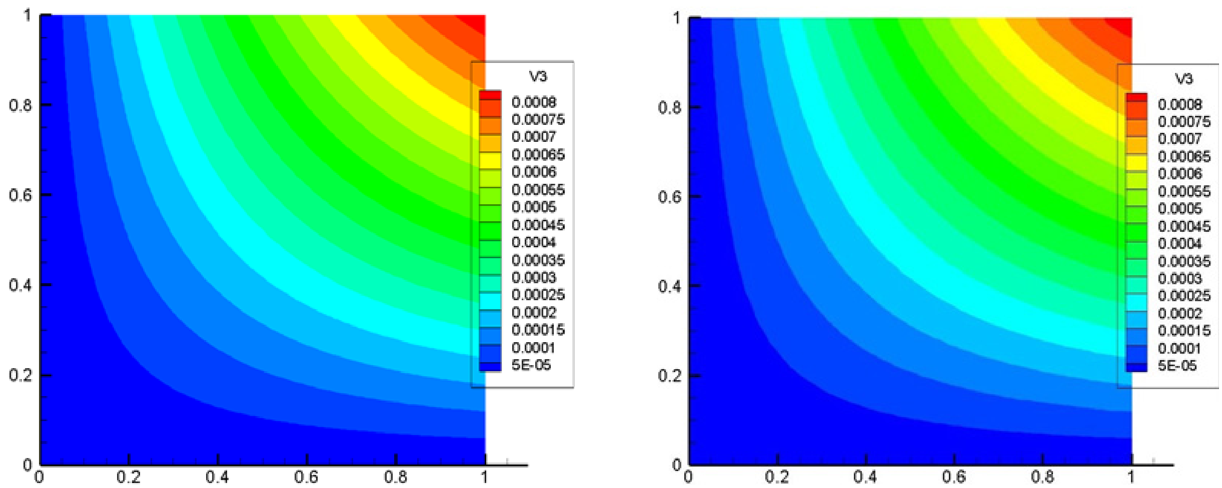

Figure 2 – Graphs of the analytical and numerical solutions in a) and b) respectively

The number of cells is 7828, the number of nodes is 4015, the time step is 10^{-10}s. The discrete form (48) has been iterated 10 000 times. The largest relative error is equal to 4.2%, but it is worth considering that this error is a single case. It follows from this that the relative error doesn't accurately represent the actual situation, which may mislead us in the future. Therefore, for this purpose, we use RMSE, similar methodology was used in [42]

$$RMSE_{FVM} = \sqrt{\frac{\sum_{i=0}^{n}(u_{inum}-u_{ian})^2}{n-1}}, \tag{49}$$

where n is the number of cells, u_{inum} is the value $u(x, y, t)$ in the numerical solution of $i-$th cell, u_{ian} is the analytical value $u(x, y, t)$ of $i-$th cell.

Calculating the error using formula (49), we obtain $RMSE_{FVM} = 9{,}9068 \cdot 10^{-7}$.

FDM

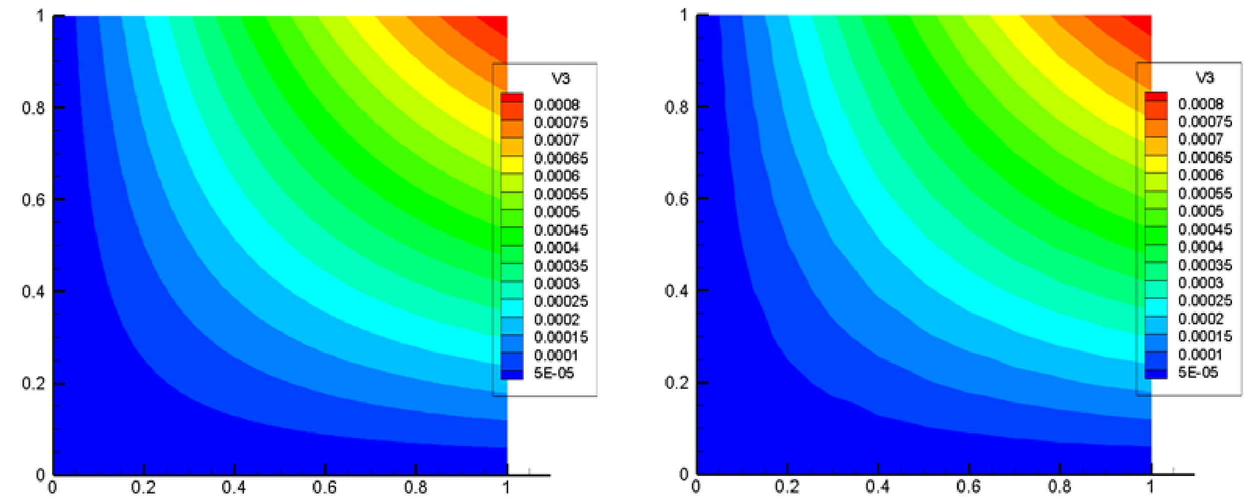

Figure 3 – Graphs of the analytical and numerical solutions in a) and b) respectively

The time step is 10^{-10}s, 10 000 iterations were performed on the obtained finite-difference scheme (14). The formula (49) was used in the FVM to estimate the error, in the case of this method, it will take the form:

$$RMSE_{FDM} = \sqrt{\frac{\sum_{i=0}^{n}(u_{inum}-u_{ian})^2}{n-1}}, \tag{50}$$

where n is the number of nodes, u_{inum} is the value of $u(x, y, t)$ in the numerical solution of $i-$th node, u_{ian} is the value of $u(x, y, t)$ in the analytical solution of $i-$th node.

The relative error is calculated by the formula

$$z^{ij} = \frac{|u(x_i y_j, t_n) - w(x_i, y_j, t_n)|}{u(x_i, y_j, t_n)}, \tag{51}$$

where $u(x, y, t)$ is the analytical solution, $w(x, y, t)$ is the numerical solution, n=10000.

Present the values of errors and computation time in seconds for different steps Δx in the form of a table.

Table 1 – Relative and RMSE

Δt	Δx	$\max_{0<i<N+1}[z^{ij}]$	δ_{FDM}	T
10^{-10}	1/10	0.000074421494	0.000000016668	19.856
	1/20	0.000074891038	0.000000017484	49.746
	1/40	0.000075008461	0.000000017794	147.799
	1/80	0.000075037819	0.000000017921	320.456
	1/100	0.000075041342	0.000000017945	405.846

We can observe that as the step size Δx decreases, the computation time also increases. The error values do not change significantly; however, with a decrease in Δx, the relative error starts to vary after $4 \cdot 10^{-7}$, and the RMSE changes after 10^{-8}. This could be attributed to the machine representation of such small numbers.

The computation time of FDM with different steps Δx is presented in Table 1, in FVM is equal to 1407.645s. Comparing the results of time and error, we can see the efficiency of FDM in our task, as the largest relative error of FVM is 4.2% , and FDM is 0.0075% , also the RMSE of the two methods indicates a smaller error in FDM, exactly $RMSE_{FDM} = 0{,}1666 \cdot 10^{-7}$ for $\Delta x = \frac{1}{10}$ and $RMSE_{FVM} = 9{,}9068 \cdot 10^{-7}$.

Conclusion

The derived stability condition $\frac{\Delta t^{\alpha}}{\Delta x^2} + \frac{\Delta t^{\alpha}}{\Delta y^2} \leq \frac{\alpha}{2}$ holds for equations with $0 < \alpha \leq 2$, meaning that in the special cases of $\alpha = 1$ and $\alpha = 2$, it serves as the stability condition for the explicit method for parabolic and hyperbolic type equations, respectively.

It is worth noticing in (15) we must store all its previous values, so FVM requires more computational resources.

The finite difference method is constructed by approximating derivatives by their discrete analogues. The advantage of the FDM is the clarity of the discretization procedure, which makes it possible to construct schemes of high order of accuracy. The disadvantage of the FDM is the limitation in the geometry of computational domains.

The algorithms were implemented using the C++ programming language.

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ПАРАБОЛАЛЫҚ ЖӘНЕ ГИПЕРБОЛАЛЫҚ ТИПТЕГІ ТЕҢДЕУЛЕР ҮШІН АҚЫРЛЫ АЙЫРЫМДЫҚ СХЕМАНЫҢ ТҰРАҚТЫЛЫҚ ШАРТЫ: БӨЛШЕК РЕТТІ ДИФФУЗИЯ ТЕҢДЕУЛЕРІ ҮШІН АҚЫРЛЫ КӨЛЕМДЕР ӘДІСІНІҢ САЛЫСТЫРЫЛУЫ

Аңдатпа

Бұл зерттеуде ақырлы айырымдар әдісі мен ақырлы көлемдер әдісінің тиімділігі салыстырмалы түрде талданады. Бұл әдістер бүтін ретті диффузия теңдеулері үшін кеңінен қолданылғанымен, уақыт бойынша бөлшек ретті диффузия теңдеулері үшін олардың орнықтылығы мен дәлдігі жеткілікті деңгейде зерттелмеген. Зерттеу барысында Грюнвальд-Летников анықтамасы негізінде бөлшек ретті туындыны жуықтау әдісі қолданылды. Ақырлы айырымдар әдісі үшін айқын айырымдық схема құрылып, уақыт бойынша бөлшек ретті айырма схемасының орнықтылық шарты қорытып шығарылды. Бұл шарт параболалық және гиперболалық теңдеулер үшін жалпылама түрде ұсынылып, уақыт бойынша бөлшек ретті схемалар үшін бұрын-соңды белгісіз болған орнықтылық критерийін анықтауға мүмкіндік берді. Сонымен қатар, ақырлы көлемдер әдісі негізінде уақыт бойынша бөлшек ретті екі өлшемді субдиффузия теңдеулерін шешуге арналған айқын дискретті схема ұсынылды. Сандық модельдеу нәтижелері көрсеткендей, ақырлы айырымдар әдісі жоғары дәлдікке ие, ал ақырлы көлемдер әдісі күрделі геометриялық пішіндер үшін неғұрлым тиімді. Алынған нәтижелер аномальды диффузия процестерін модельдеуде сандық әдістерді жетілдіруге маңызды негіз бола алады.

Тірек сөздер: субдиффузия, ақырлы айырымдар әдісі, ақырлы көлем әдісі, Грюнвальд-Летников бөлшек ретті туындысы, тұрақтылық шарты.

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УСЛОВИЕ УСТОЙЧИВОСТИ КОНЕЧНО-РАЗНОСТНОЙ СХЕМЫ ДЛЯ УРАВНЕНИЙ ПАРАБОЛИЧЕСКОГО И ГИПЕРБОЛИЧЕСКОГО ТИПОВ: СРАВНЕНИЕ МЕТОДА КОНЕЧНЫХ ОБЪЕМОВ ДЛЯ УРАВНЕНИЙ ДИФФУЗИИ ДРОБНОГО ПОРЯДКА

Аннотация

В данной работе проведен сравнительный анализ методов конечных разностей и объемов. Данные методы широко известны для уравнений диффузии с целым порядком, но тем не менее недостаточно исследована эффективность данных методов для уравнений диффузии с дробным порядком по времени. Для аппроксимации уравнения использовано определение дробной производной Грюнвальда-Летникова. Для метода конечных разностей получена явная разностная схема и выведено условие устойчивости для разностной схемы с дробным порядком по времени, что также является обобщением для параболических и гиперболических типов уравнений, что ранее было неизвестно для схем с дробным порядком по времени. Представлена явная дискретная форма для решения уравнений субдиффузии в двумерном пространстве с дробным порядком по времени методом конечных объемов. Результаты показывают, что метод конечных разностей демонстрирует высокую точность, тогда как метод конечных объемов лучше подходит для сложных геометрических форм. Эти результаты открывают возможности для дальнейшего развития численных методов в задачах, связанных с моделированием процессов аномальной диффузии.

Ключевые слова: субдиффузия, метод конечных разностей, метод конечных объемов, дробная производная Грюнвальда-Летникова, условие устойчивости.

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