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HARDY INEQUALITIES AND IDENTITIES RELATED TO THE BAOUENDI-GRUSHIN VECTOR FIELDS AND LANDAU-HAMILTONIAN

Abstract

In this paper, we present a weighted Hardy identity related to the Baouendi-Grushin vector fields and its applications in the context of differential inequalities. By selecting appropriate parameters, the Hardy identity related to the Baouendi-Grushin operator implies numerous sharp remainder formulae for Hardy type inequalities. In the commutative case, we obtain improved weighted Hardy inequalities in the setting of the Euclidean space. For example, in a special case, by dropping non-negative remainder terms, related to the Baouendi-Grushin operator, and choosing suitable parameters our identity allows us to derive an improved critical Hardy inequality for the radial derivative operator with a sharp constant that does not depend on the topological dimension. We employ the method of factorizing differential expressions, as used by Gesztesy and Littlejohn in [1]. In this paper, we demonstrate the application of the factorization method in the noncommutative Baouendi-Grushin setting. As an application of the Hardy identity related to the Baouendi-Grushin vector fields, we establish a Hardy inequality for the generalized Landau Hamiltonian (or the twisted Laplacian) with remainder terms.

Key words: factorization method, Hardy inequality, Baouendi-Grushin operator, Landau-Hamiltonian.

Introduction

The classical Hardy inequality for functions $f \in C_0^\infty(\mathbb{R}^n\setminus\{0\})$ is

$$\int_{\mathbb{R}^n} |\nabla f(z)|^2 \, dz \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|f(z)|^2}{|z|^2} dz, \quad n \geq 3, \tag{1}$$

where the constant $\left(\frac{n-2}{2}\right)^2$ is sharp but not attained. Originating from Hardy's work [2] in 1920, initially focusing on the one-dimensional case, this inequality has since garnered significant attention from mathematicians and undergone extensive analysis in various directions. The enormous amount of research that has been done through the years make it unachievable to list all the relevant references on this subject. Therefore, we only refer to the standard monographs such as [3–8].

Baouendi-Grushin operator

The sharp Hardy inequality (1) arises very naturally in the study of degenerate elliptic differential operators and it was first extended in [9] by Garofalo to the Baouendi-Grushin vector fields

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \dots, m, \quad Y_j = |x|^\gamma \frac{\partial}{\partial y_j}, \quad j = 1, \dots, k,$$

where $x \in \mathbb{R}^m, y \in \mathbb{R}^k$ with $m, k \geq 1, \gamma \geq 0$. To be explicit, Garofalo in [9] proved the following Hardy inequality

$$\int_{\mathbb{R}^n} \left(|(\nabla_x f)(z)|^2 + |x|^{2\gamma}\left|(\nabla_y f)(z)\right|^2\right) dz$$

$$\geq \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|x|^{2\gamma}}{|x|^{2+2\gamma}+(1+\gamma)^2|y|^2} |f(z)|^2 dz, \qquad (2)$$

for every $f \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^k \backslash \{(0,0)\})$ with $n = m + k$, $Q = m + (1+\gamma)k$. Here, $(\nabla_x f)(z)$ and $(\nabla_y f)(z)$ are the gradients of f in the variables x and y respectively. Moreover, the inequality (2) recovers (1) if $\gamma = 0$.

The corresponding sub-elliptic gradient to the Baouendi-Grushin vector fields is defined as

$$\nabla_\gamma := (X_1, \dots, X_m, Y_1, \dots, Y_k) = \left(\nabla_x, |x|^\gamma \nabla_y\right).$$

The Baouendi-Grushin operator on $\mathbb{R}^n$ is

$$\Delta_\gamma = \sum_{i=1}^m X_i^2 + \sum_{j=1}^k Y_j^2 = \Delta_x + |x|^{2\gamma}\Delta_y = \nabla_\gamma \cdot \nabla_\gamma,$$

where Δ_x and Δ_y are the Laplacians on $\mathbb{R}^m$ and $\mathbb{R}^k$, respectively. The Baouendi-Grushin operator is a sum of squares of C^∞ vector fields satisfying the Hörmander condition for even positive integers γ

$$\text{rank Lie}[X_1, \dots, X_m, Y_1, \dots, Y_k] = n.$$

We can define on $\mathbb{R}^{m+k}$ the anisotropic dilation attached to Δ_γ as

$$\delta_\lambda(x,y) = (\lambda x, \lambda^{1+\gamma} y),$$

for $\lambda > 0$, and the homogeneous dimension with respect to this dilation is

$$Q = m + (1+\gamma)k.$$

A change of variables formula for the Lebesgue measure implies that

$$d \circ \delta_\lambda(x,y) = \lambda^Q dxdy.$$

It is easy to check that

$$X_i(\delta_\lambda) = \lambda\delta_\lambda(X_i), \quad Y_i(\delta_\lambda) = \lambda\delta_\lambda(Y_i),$$

and hence

$$\nabla_\gamma \circ \delta_\lambda = \lambda\delta_\lambda \nabla_\gamma.$$

Let $\rho(z)$ be the corresponding distance function from the origin for $z = (x,y) \in \mathbb{R}^n$:

$$\rho = \rho(z) := \left(|x|^{2(1+\gamma)} + (1+\gamma)^2|y|^2\right)^{\frac{1}{2(1+\gamma)}}.$$

Material and methods

The main purpose of this paper is to derive a weighted Hardy identity for Baouendi-Grushin vector fields via the method of factorization. Initially, Gesztesy used the method with regards to the classical Hardy inequality [10] and its logarithmic refinements [11]. In short, let us demonstrate this method: for any given $n \in \mathbb{N}$, $n \geq 3$, let us consider the following vector-valued differential expression

$$T := \nabla + \frac{n-2}{2}|x|^{-2}x, \quad x \in \mathbb{R}^n \backslash \{0\},$$

with the formal adjoint

$$T^+ = -div(\cdot) + \frac{n-2}{2}|x|^{-2}x\,\cdot, \quad x \in \mathbb{R}^n \backslash \{0\},$$

such that,

$$T^+T = -\Delta + \frac{n-2}{2}\left(\frac{n-2}{2} + 2 - n\right)|x|^{-2}.$$

Then, using the non-negativity of $T^{+}T$, for $f \in C_0^{\infty}(\mathbb{R}^n\backslash\{0\})$, one gets

$$0 \leq \int_{\mathbb{R}^n} |(Tf)(x)|^2 dx = \int_{\mathbb{R}^n} \overline{f(x)}\,(T^{+}Tf)(x)dx$$
$$= \int_{\mathbb{R}^n} |(\nabla f)(x)|^2 dx + \frac{n-2}{2}\left(\frac{n-2}{2} + 2 - n\right)\int_{\mathbb{R}^n} |x|^{-2}|f(x)|^2 dx.$$

As a result, one obtains the following inequality with the sharp constant:

$$\int_{\mathbb{R}^n} |(\nabla f)(x)|^2\, dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2} dx, \quad f \in C_0^{\infty}(\mathbb{R}^n\backslash\{0\}), \quad n \geq 3.$$

The method was then employed in [1] to demonstrate how factorization of singular, even-order partial differential operators give elementary approach to classical inequalities of Hardy–Rellich-type. Lam, Lu and Zhang [12] applied factorizations to obtain Hardy's type identities and inequalities on upper half spaces. Furthermore, Ruzhansky and Yessirkegenov [13], using the same technique, were able to derive Hardy, weighted Hardy, improved Hardy and Hardy-Rellich inequalities on stratified, on Heisenberg and on homogeneous groups.

Results and Discussion

Theorem 1. Let $\alpha, a, b, c, d \in \mathbb{R}$. Then the identity

$$\left\| (log\,\rho)^c \frac{z \cdot (\nabla_\gamma f)(z)}{\rho^{a+\gamma+1}} + \alpha \frac{(log\,\rho)^d}{\rho^b} f(z) \right\|^2_{L^2(\mathbb{R}^n)}$$

$$= \int_{\mathbb{R}^n} (\rho^{-a}(log\,\rho)^c)^2 \left| \frac{z \cdot (\nabla_\gamma f)(z)}{\rho^{\gamma+1}} \right|^2 dz$$

$$+\alpha \int_{\mathbb{R}^n} \left[\frac{(\log \rho)^{c+d}\left(|x|^{2\gamma+2} + |x|^{\gamma(1+\gamma)}|y|^2\right)}{\rho^{a+b+3\gamma+3}} \left(a + b + 1 + \gamma - \frac{c+d}{\log \rho}\right) \right] |f(z)|^2\, dz$$

$$+\alpha \int_{\mathbb{R}^n} \left[-\frac{m + k|x|^\gamma}{\rho^{a+b+1+\gamma}} (log\,\rho)^{c+d} + \alpha \frac{(log\,\rho)^{2d}}{\rho^{2b}} \right] |f(z)|^2\, dz, \tag{3}$$

holds for every $f \in C_0^{\infty}(\mathbb{R}^n\backslash\{0\})$.

$$\left\| (\log|z|)^c \frac{\partial_{|z|} f(z)}{|z|^a} + \alpha \frac{(\log|z|)^d}{|z|^b} f(z) \right\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \frac{(\log|z|)^{2c}}{|z|^{2a}} \left|\partial_{|z|} f(z)\right|^2 dz$$

$$-\alpha \int_{\mathbb{R}^n} \left[\frac{(\log|z|)^{c+d}}{|z|^{a+b+1}} \left(\frac{c+d}{\log|z|} + n - (a+b+1) \right) \right] |f(z)|^2\, dz$$

$$+\alpha^2 \int_{\mathbb{R}^n} \frac{(\log|z|)^{2d}}{|z|^{2b}} |f(z)|^2\, dz, \tag{4}$$

Remark 1. If $\gamma = 0$ in (3), then the following identity is obtained:
where, $\partial_{|z|} f = \frac{\partial f(z)}{\partial |z|} = \frac{z\cdot(\nabla f)(z)}{|z|}$ is the radial derivative of f with respect to $z = (x, y)$. Now after dropping the left-hand side of (4), specifying the parameters ($a = \frac{n-2}{2}$, $b = \frac{n}{2}$, $c = 1$ and $d = 0$), and maximising the constant with respect to α, we derive the following critical Hardy inequality in $\mathbb{R}^n$ from [Formula 3.6, 13]:

$$\int_{\mathbb{R}^n} \frac{(\log|z|)^2}{|z|^{n-2}} \left|\partial_{|z|} f(z)\right|^2 dz \geq \frac{1}{4}\int_{\mathbb{R}^n} \frac{|f(z)|^2}{|z|^n} dz. \tag{5}$$

For the sharpness of the constant in (5) we refer to [14]. We also refer to [15, 16, 17, 18, 19] for such inequalities on homogeneous Lie groups.

Let us now demonstrate an application of the obtained result in the magnetic inequality related to the Landau-Hamiltonian type magnetic field. First, let us recall from [20] the generalized twisted Laplacian with Landau-Hamiltonian type magnetic field defined by

$$\mathcal{L} = \sum_{j=1}^{n} \left[\left(i\,\partial_{x_j} + \psi(|z|)y_j\right)^2 + \left(i\,\partial_{y_j} - \psi(|z|)x_j\right)^2\right],$$

where $\psi(|z|)$ is a radial real-valued differentiable function. Denoting

$$\tilde{X}_j = \partial_{x_j} - i\psi(|z|)y_j, \text{ and } \tilde{Y}_j = \partial_{y_j} + i\psi(|z|)x_j,$$

we write

$$\widetilde{\nabla_{\mathcal{L}}} f = \left(\tilde{X}_1 f, \dots, \tilde{X}_n f, \tilde{Y}_1 f, \dots, \tilde{Y}_n f\right).$$

Corollary 1. Let $\alpha, a, b, c, d \in \mathbb{R}$. Let $\psi = \psi(|z|)$ be a radial real-valued function such that $\psi \in L^2_{loc}(\mathbb{C}\backslash\{0\})$ with $|\psi(r)|r^2 \leq 1/2$, $\forall r \in (0, \infty)$. Then for every $f \in C_0^\infty(\mathbb{C}\backslash\{0\})$ we have where $\mathbb{C}$ is the set of complex numbers.

Proof of Corollary 1. We will use the following inequality from [Formula 44, 20]:

$$\int_{\mathbb{C}} (|z|^{-a}(log|z|)^c)^2 \left|\widetilde{\nabla_{\mathcal{L}}} f\right|^2 dz \geq \int_{\mathbb{C}} (|z|^{-a+1}(log|z|)^c)^2 \psi(|z|)^2 |f(z)|^2 dz$$

$$+ \left\|\frac{(log\,|z|)^c}{|z|^a}\frac{df}{d|z|} + \alpha\frac{(log\,|z|)^d}{|z|^b} f(z)\right\|^2_{L^2(\mathbb{C})} - \alpha\int_{\mathbb{C}}\left[\frac{(log\,|z|)^{c+d}}{|z|^{a+b+1}}\left(a+b+1-\frac{c+d}{log\,|z|}\right) - \frac{m+k}{|z|^{a+b+1}}(log|z|)^{c+d} + \alpha\frac{(log\,|z|)^{2d}}{|z|^{2b}}\right]|f(z)|^2\,dz,$$

$$\int_{\mathbb{C}} \frac{\left|\widetilde{\nabla_{\mathcal{L}}} f\right|^2}{\kappa(|z|)} dz \geq \int_{\mathbb{C}} \frac{1}{\kappa(|z|)}\left|\frac{d}{d|z|}f\right|^2 dz + \int_{\mathbb{C}} \frac{|z|^2\psi(|z|)^2}{\kappa(|z|)}|f|^2 dz$$

for all $f \in C_0^\infty(\mathbb{C}\backslash\{0\})$ and for every radial function $\kappa(|z|) \neq 0$ with $\kappa(|z|)^{-1} \in L^2_{\mathrm{loc}}(\mathbb{C}\setminus\{0\})$. Taking $\kappa(|z|) = (|z|^{-a}(log\,|z|)^c)^{-2}$ and substituting (3) when $\gamma = 0$ into the above inequality concludes the proof.

Proof of Theorem 1. Let us first introduce the six-parameter differential expression

$$T_{\alpha,\gamma,a,b,c,d} := \rho^{-a}(\log\rho)^c \frac{z\cdot\nabla_\gamma}{\rho^{\gamma+1}} + \alpha\rho^{-b}(\log\rho)^d. \tag{6}$$

Calculating the formal adjoint:

$$\left\langle \left(T_{\alpha,\gamma,a,b,c,d} f\right)(z), \overline{g(z)} \right\rangle = \int_{\mathbb{R}^n} \left(\rho^{-a} (\log\rho)^c \frac{z \cdot \left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}} + \alpha\rho^{-b} (\log\rho)^d f(z) \right) \overline{g(z)}\, dz$$

$$= \int_{\mathbb{R}^n} \frac{\overline{g(z)}\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}} \left(\sum_{i=1}^{m} x_i\, \partial_{x_i}\left(f(z)\right) + |x|^\gamma \sum_{j=1}^{k} y_j\, \partial_{y_j}\left(f(z)\right) \right) dz + \alpha \int_{\mathbb{R}^n} \rho^{-b} (\log\rho)^d f(z)\overline{g(z)}$$

$$= \sum_{i=1}^{m} \int_{\mathbb{R}^n} \frac{\overline{g(z)}\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}} x_i\, \partial_{x_i}\left(f(z)\right) dz + \sum_{j=1}^{k} \int_{\mathbb{R}^n} \frac{\overline{g(z)}\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}} |x|^\gamma\, y_j\, \partial_{y_j}\left(f(z)\right) dz$$

$$+\alpha \int_{\mathbb{R}^n} \rho^{-b} (\log\rho)^d f(z)\overline{g(z)} dz$$

$$= -\sum_{i=1}^{m} \int_{\mathbb{R}^n} \partial_{x_i} \left(\frac{\overline{g(z)}\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}} \right) x_i\, f(z) dz - m \int_{\mathbb{R}^n} \frac{\overline{g(z)}\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}} f(z) dz$$

$$-\sum_{j=1}^{k} \int_{\mathbb{R}^n} \partial_{y_j} \left(\frac{\overline{g(z)}\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}} \right) |x|^\gamma\, y_j f(z) dz - k \int_{\mathbb{R}^n} \frac{\overline{g(z)}\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}} |x|^\gamma f(z) dz$$

$$+\alpha \int_{\mathbb{R}^n} \rho^{-b} (\log\rho)^d f(z)\overline{g(z)} dz.$$

Since $\partial_{x_i}(\rho^{\gamma+1}) = (\gamma+1)\rho^{-\gamma-1}|x|^{2\gamma}x_i$ and $\partial_{y_j}(\rho^{\gamma+1}) = (\gamma+1)^2\rho^{-\gamma-1}y_j$, we obtain

$$\left\langle \left(T_{\alpha,\gamma,a,b,c,d} f\right)(z), \overline{g(z)} \right\rangle =$$

$$-\sum_{i=1}^{m} \int_{\mathbb{R}^n} \left(\frac{-\overline{g(z)}\rho^{-a}(\log\rho)^c \frac{(1+\gamma)x_i|x|^{2\gamma}}{\rho^{\gamma+1}}}{\rho^{2\gamma+2}} \right) x_i\, f(z) dz$$

$$-\sum_{i=1}^{m} \int_{\mathbb{R}^n} \left(\frac{\overline{g(z)}\, \partial_{x_i}(\rho^{-a}(\log\rho)^c)}{\rho^{\gamma+1}} \right) x_i\, f(z) dz$$

$$-\sum_{i=1}^{m} \int_{\mathbb{R}^n} \left(\frac{\rho^{-a}(log\,\rho)^c\, \partial_{x_i}\left(\overline{g(z)}\right)}{\rho^{\gamma+1}} \right) x_i f(z) dz$$

$$-\int_{\mathbb{R}^n} \frac{\overline{g(z)}\rho^{-a}(\log\rho)^c f(z)}{\rho^{\gamma+1}} (m + k|x|^\gamma) dz$$

$$-\sum_{j=1}^{k}\int_{\mathbb{R}^n}\left(\frac{-\overline{g(z)}\rho^{-a}(\log\rho)^c\frac{y_j(1+\gamma)^2}{\rho^{\gamma+1}}}{\rho^{2\gamma+2}}\right)|x|^\gamma y_j f(z)dz$$

$$-\sum_{j=1}^{k}\int_{\mathbb{R}^n}\left(\frac{\overline{g(z)}\,\partial_{y_j}(\rho^{-a}(\log\rho)^c)}{\rho^{\gamma+1}}\right)|x|^\gamma y_j f(z)dz$$

$$-\sum_{j=1}^{k}\int_{\mathbb{R}^n}\left(\frac{\rho^{-a}(log\,\rho)^c\,\partial_{y_j}\left(\overline{g(z)}\right)}{\rho^{\gamma+1}}\right)|x|^\gamma y_j f(z)dz+\alpha\int_{\mathbb{R}^n}\rho^{-b}(\log\rho)^d f(z)\overline{g(z)}dz.$$

Now we open the brackets:

$$\left\langle\left(T_{\alpha,\gamma,a,b,c,d}f\right)(z),\overline{g(z)}\right\rangle=\sum_{i=1}^{m}\int_{\mathbb{R}^n}\frac{f(z)\overline{g(z)}\rho^{-a}(\log\rho)^c\frac{(1+\gamma)x_i^2|x|^{2\gamma}}{\rho^{\gamma+1}}}{\rho^{2\gamma+2}}dz$$

$$-\sum_{i=1}^{m}\int_{\mathbb{R}^n}\frac{x_i f(z)\overline{g(z)}\,\partial_{x_i}(\rho^{-a}(\log\rho)^c)}{\rho^{\gamma+1}}dz$$

$$-\sum_{i=1}^{m}\int_{\mathbb{R}^n}\frac{x_i f(z)\rho^{-a}(\log\rho)^c\,\partial_{x_i}\overline{g(z)}}{\rho^{\gamma+1}}dz-\int_{\mathbb{R}^n}\frac{\overline{g(z)}\rho^{-a}(\log\rho)^c f(z)}{\rho^{\gamma+1}}(m+k|x|^\gamma)dz$$

$$+\sum_{j=1}^{k}\int_{\mathbb{R}^n}|x|^\gamma\frac{f(z)\overline{g(z)}\rho^{-a}(\log\rho)^c\frac{y_j^2(1+\gamma)^2}{\rho^{\gamma+1}}}{\rho^{2\gamma+2}}dz-\sum_{j=1}^{k}\int_{\mathbb{R}^n}|x|^\gamma\frac{y_i f(z)\overline{g(z)}\,\partial_{y_j}(\rho^{-a}(\log\rho)^c)}{\rho^{\gamma+1}}dz$$

$$-\sum_{j=1}^{k}\int_{\mathbb{R}^n}|x|^\gamma\frac{y_j f(z)\rho^{-a}(\log\rho)^c\,\partial_{y_j}\overline{g(z)}}{\rho^{\gamma+1}}dz+\alpha\int_{\mathbb{R}^n}\rho^{-b}(\log\rho)^d f(z)\overline{g(z)}dz.$$

Simplifying, we get the following:

$$\left\langle\left(T_{\alpha,\gamma,a,b,c,d}f\right)(z),\overline{g(z)}\right\rangle$$

$$=\int_{\mathbb{R}^n}\frac{f(z)\overline{g(z)}\rho^{-a}(\log\rho)^c\left((1+\gamma)|x|^{2\gamma+2}+|x|^\gamma(1+\gamma)^2|y|^2\right)}{\rho^{3\gamma+3}}dz$$

$$-\int_{\mathbb{R}^n}f(z)\overline{g(z)}\frac{z\cdot\nabla_\gamma\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}dz$$

$$-\int_{\mathbb{R}^n}f(z)\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma\overline{g}\right)(z)}{\rho^{\gamma+1}}dz-\int_{\mathbb{R}^n}\frac{\overline{g(z)}\rho^{-a}(\log\rho)^c f(z)}{\rho^{\gamma+1}}(m+k|x|^\gamma)dz$$

$$+\alpha\int_{\mathbb{R}^n}\rho^{-b}(\log\rho)^d f(z)\overline{g(z)}dz$$

$$= \int_{\mathbb{R}^n} f(z)\left(\frac{\overline{g(z)}\rho^{-a}(\log\rho)^c\left((1+\gamma)|x|^{2\gamma+2}+|x|^\gamma(1+\gamma)^2|y|^2\right)}{\rho^{3\gamma+3}} - \overline{g(z)}\frac{z\cdot\nabla_\gamma\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}\right)dz$$

$$+\int_{\mathbb{R}^n} f(z)\left(-\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma\overline{g}\right)(z)}{\rho^{\gamma+1}} - \frac{\overline{g(z)}\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}(m+k|x|^\gamma) + \alpha\rho^{-b}(\log\rho)^d\overline{g(z)}\right)dz$$

$$= \int_{\mathbb{R}^n} f(z)\left(-\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma\overline{g}\right)(z)}{\rho^{\gamma+1}}\right)dz$$

$$+\int_{\mathbb{R}^n} f(z)\left(-\left(\frac{m+k|x|^\gamma}{\rho^{\gamma+1}} - \frac{(1+\gamma)|x|^{2\gamma+2}+|x|^\gamma(1+\gamma)^2|y|^2}{\rho^{3\gamma+3}}\right)\rho^{-a}(\log\rho)^c\overline{g(z)} - \frac{z\cdot\nabla_\gamma\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}\overline{g(z)}\right)dz$$

$$+\alpha\int_{\mathbb{R}^n} f(z)\rho^{-b}(\log\rho)^d\overline{g(z)}\,dz.$$

Thus, the formal adjoint operator of $T_{\alpha,\gamma,a,b,c,d}$ is the following:

$$T^+_{\alpha,\gamma,a,b,c,d} = -\rho^{-a}(\log\rho)^c\frac{z\cdot\nabla_\gamma}{\rho^{\gamma+1}} - \left(\frac{m+k|x|^\gamma}{\rho^{\gamma+1}} - \frac{(1+\gamma)|x|^{2\gamma+2}+|x|^\gamma(1+\gamma)^2|y|^2}{\rho^{3\gamma+3}}\right)\rho^{-a}(\log\rho)^c$$

$$-\frac{z\cdot\nabla_\gamma\rho^{-a}(log\,\rho)^c}{\rho^{\gamma+1}} + \alpha\rho^{-b}(\log\rho)^d.$$

Then, we calculate $(T^+{}_{\alpha,\gamma,a,b,c,d}T_{\alpha,\gamma,a,b,c,d}f)(z)$:

$$(T^+{}_{\alpha,\gamma,a,b,c,d}T_{\alpha,\gamma,a,b,c,d}f)(z) = T^+{}_{\alpha,\gamma,a,b,c,d}\left(\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}} + \alpha\rho^{-b}(\log\rho)^d f(z)\right)$$

$$= I_1 + I_2 + I_3 + I_4, \tag{7}$$

where

$$I_1 = -\frac{\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}\left(z\cdot\nabla_\gamma\left(\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}} + \alpha\rho^{-b}(\log\rho)^d f(z)\right)\right),$$

$$I_2 = -\left(\frac{m+k|x|^\gamma}{\rho^{\gamma+1}} - \frac{(1+\gamma)|x|^{2\gamma+2}+|x|^\gamma(1+\gamma)^2|y|^2}{\rho^{3\gamma+3}}\right)\left((\rho^{-a}(\log\rho)^c)^2\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}\right)$$

$$-\alpha\left(\frac{m+k|x|^\gamma}{\rho^{\gamma+1}} - \frac{(1+\gamma)|x|^{2\gamma+2}+|x|^\gamma(1+\gamma)^2|y|^2}{\rho^{3\gamma+3}}\right)\rho^{-a}(\log\rho)^c\rho^{-b}(\log\rho)^d f(z),$$

$$I_3 = -\frac{z\cdot\nabla_\gamma\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}\left(\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}} + \alpha\rho^{-b}(\log\rho)^d f(z)\right),$$

$$-\int_{\mathbb{R}^n}\frac{\rho^{-a}(log\,\rho)^c}{\rho^{\gamma+1}}\left[|x|^\gamma\sum_{j=1}^{k}y_j\,\partial_{y_j}\left(\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}\right)\right]\overline{f(z)}\,dz, \tag{13}$$

$$U_2=-\alpha\int_{\mathbb{R}^n}\overline{f(z)}\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma\rho^{-b}(log\,\rho)^d f\right)(z)}{\rho^{\gamma+1}}dz, \tag{14}$$

$$U_3=-\int_{\mathbb{R}^n}\overline{f(z)}\left(\frac{m+k|x|^\gamma}{\rho^{\gamma+1}}-\frac{(1+\gamma)|x|^{2\gamma+2}+|x|^\gamma(1+\gamma)^2|y|^2}{\rho^{3\gamma+3}}\right)$$

$$\times\left((\rho^{-a}(\log\rho)^c)^2\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}\right)dz, \tag{15}$$

$$U_4=-\alpha\int_{\mathbb{R}^n}\overline{f(z)}\left(\frac{m+k|x|^\gamma}{\rho^{\gamma+1}}-\frac{(1+\gamma)|x|^{2\gamma+2}+|x|^\gamma(1+\gamma)^2|y|^2}{\rho^{3\gamma+3}}\right)$$

$$\times\,\rho^{-a}(\log\rho)^c\rho^{-b}(\log\rho)^d f(z)dz, \tag{16}$$

$$U_5=-\int_{\mathbb{R}^n}\rho^{-a}(\log\rho)^c\overline{f(z)}\frac{z\cdot\nabla_\gamma\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}dz, \tag{17}$$

$$U_6=-\alpha\int_{\mathbb{R}^n}\overline{f(z)}\,\rho^{-b}(\log\rho)^d f(z)\frac{z\cdot\nabla_\gamma\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}dz, \tag{18}$$

$$U_7=\alpha\int_{\mathbb{R}^n}\overline{f(z)}\,\rho^{-b}(\log\rho)^d\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}dz, \tag{19}$$

$$U_8=\alpha^2\int_{\mathbb{R}^n}\overline{f(z)}\,(\rho^{-b}(\log\rho)^d)^2 f(z)dz. \tag{20}$$

First, we calculate (13):

$$U_1=-\int_{\mathbb{R}^n}\frac{\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}\left[\sum_{i=1}^{m}x_i\,\partial_{x_i}\left(\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}\right)\right.$$

$$\left.+|x|^\gamma\sum_{j=1}^{k}y_j\,\partial_{y_j}\left(\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}\right)\right]\overline{f(z)}\,dz$$

$$=A_1+\widetilde{A_1}. \tag{21}$$

For A_1, we have

$$A_1=-\int_{\mathbb{R}^n}\frac{\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}\left(\sum_{i=1}^{m}x_i\,\partial_{x_i}\left(\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}\right)\right)\overline{f(z)}\,dz$$

$$=-\int_{\mathbb{R}^n}\frac{\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}\left[\sum_{i=1}^{m}x_i\left[\rho^{-a}(\log\rho)^c\,\partial_{x_i}\left(\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}\right)+\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}\partial_{x_i}(\rho^{-a}(\log\rho)^c)\right]\right]\overline{f(z)}\,dz$$

$$= -\int_{\mathbb{R}^n} \overline{f(z)} \frac{\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}} \sum_{i=1}^{m} x_i\, \rho^{-a}(\log\rho)^c\, \partial_{x_i}\left(\frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}}\right) dz$$

$$-\int_{\mathbb{R}^n} \overline{f(z)} \frac{\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}} \sum_{i=1}^{m} x_i \frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}} \partial_{x_i}(\rho^{-a}(\log\rho)^c) dz$$

$$= -\sum_{i=1}^{m}\int_{\mathbb{R}^n} \overline{f(z)} \frac{\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}} x_i\, \rho^{-a}(\log\rho)^c\, \partial_{x_i}\left(\frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}}\right) dz$$

$$-\sum_{i=1}^{m}\int_{\mathbb{R}^n} \overline{f(z)} \frac{\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}} x_i \frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}} \partial_{x_i}(\rho^{-a}(\log\rho)^c) dz$$

$$= -\sum_{i=1}^{m}\int_{\mathbb{R}^n} \overline{f(z)} \frac{(\rho^{-a}(\log\rho)^c)^2}{\rho^{\gamma+1}} x_i\, \partial_{x_i}\left(\frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}}\right) dz$$

$$-\sum_{i=1}^{m}\int_{\mathbb{R}^n} \overline{f(z)} \frac{\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}} x_i \frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}} \partial_{x_i}(\rho^{-a}(\log\rho)^c) dz$$

$$= B_1 + B_2. \tag{22}$$

We will consider B_2 first. It will become clear why we did so later.

$$B_2 = -\sum_{i=1}^{m}\int_{\mathbb{R}^n} \overline{f(z)} \frac{\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}} x_i \frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}} \partial_{x_i}(\rho^{-a}(\log\rho)^c) dz$$

$$= \sum_{i=1}^{m}\int_{\mathbb{R}^n} \partial_{x_i}\left(\overline{f(z)}\right) \frac{(\rho^{-a}(\log\rho)^c)^2}{\rho^{\gamma+1}} x_i \frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}} dz$$

$$+\sum_{i=1}^{m}\int_{\mathbb{R}^n} \overline{f(z)}\, \partial_{x_i}\left(\frac{\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}\right) x_i \frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}} \rho^{-a}(\log\rho)^c dz$$

$$+m\int_{\mathbb{R}^n} \overline{f(z)} \frac{(\rho^{-a}(\log\rho)^c)^2}{\rho^{\gamma+1}} \frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}} dz$$

$$+\sum_{i=1}^{m}\int_{\mathbb{R}^n} \overline{f(z)} \frac{(\rho^{-a}(\log\rho)^c)^2}{\rho^{\gamma+1}} x_i\, \partial_{x_i}\left(\frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}}\right) dz$$

$$= C_1 + C_2 + C_3 + C_4. \tag{23}$$

For C_1:

$$C_1 = \sum_{i=1}^{m}\int_{\mathbb{R}^n} \partial_{x_i}\left(\overline{f(z)}\right) \frac{(\rho^{-a}(\log\rho)^c)^2}{\rho^{\gamma+1}} x_i \frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}} dz. \tag{24}$$

For C_2:

$$C_2=\sum_{i=1}^{m}\int_{\mathbb{R}^n}\overline{f(z)}\,\partial_{x_i}\left(\frac{\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}\right)x_i\frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}}\rho^{-a}(\log\rho)^c dz$$

$$=\sum_{i=1}^{m}\int_{\mathbb{R}^n}\overline{f(z)}\frac{-\rho^{-a}(\log\rho)^c(1+\gamma)\rho^{-\gamma-1}|x|^{2\gamma}x_i+\rho^{1+\gamma}\,\partial_{x_i}(\rho^{-a}(\log\rho)^c)}{\rho^{2\gamma+2}}$$

$$\times x_i\frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}}\rho^{-a}(\log\rho)^c dz$$

$$=-\int_{\mathbb{R}^n}\overline{f(z)}\frac{(\rho^{-a}(\log\rho)^c)^2(1+\gamma)|x|^{2\gamma+2}}{\rho^{3\gamma+3}}\frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}}dz$$

$$+\sum_{i=1}^{m}\int_{\mathbb{R}^n}\rho^{-a}(\log\rho)^c\overline{f(z)}\frac{\partial_{x_i}(\rho^{-a}(\log\rho)^c)}{\rho^{\gamma+1}}x_i\frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}}dz. \tag{25}$$

For C_3:

$$C_3=m\int_{\mathbb{R}^n}\overline{f(z)}\frac{(\rho^{-a}(\log\rho)^c)^2}{\rho^{\gamma+1}}\frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}}dz. \tag{26}$$

For C_4:

$$C_4=\sum_{i=1}^{m}\int_{\mathbb{R}^n}\overline{f(z)}\frac{(\rho^{-a}(\log\rho)^c)^2}{\rho^{\gamma+1}}x_i\,\partial_{x_i}\left(\frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}}\right)dz. \tag{27}$$

Putting (24)-(27) to (23), and then (23) to (22) we get

$$A_1=B_1+B_2$$

$$=B_1+C_1+C_2+C_3+C_4. \tag{28}$$

Since $C_4=-B_1$, (28) implies that

$$A_1=C_1+C_2+C_3. \tag{29}$$

The calculations for $\widetilde{A_1}$ are analogous:

$$\widetilde{A_1}=\widetilde{C_1}+\widetilde{C_2}+\widetilde{C_3}, \tag{30}$$

where

$$\widetilde{C_1}=\sum_{j=1}^{k}\int_{\mathbb{R}^n}|x|^\gamma\,\partial_{y_i}\left(\overline{f(z)}\right)\frac{(\rho^{-a}(\log\rho)^c)^2}{\rho^{\gamma+1}}y_j\frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}}dz,$$

$$\widetilde{C_2}=-\int_{\mathbb{R}^n}\overline{f(z)}\frac{(\rho^{-a}(\log\rho)^c)^2|x|^\gamma|y|^2(1+\gamma)^2}{\rho^{3\gamma+3}}\frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}}dz$$

$$+\sum_{j=1}^{k}\int_{\mathbb{R}^n}|x|^\gamma\rho^{-a}(\log\rho)^c\overline{f(z)}\frac{\partial_{y_j}(\rho^{-a}(\log\rho)^c)}{\rho^{\gamma+1}}y_j\frac{z\cdot(\nabla_\gamma f)(z)}{\rho^{\gamma+1}}dz,$$

$$\widetilde{C_3} = k\int_{\mathbb{R}^n} |x|^{\gamma}\overline{f(z)}\frac{(\rho^{-a}(\log\rho)^c)^2}{\rho^{\gamma+1}}\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}dz.$$

Therefore, substituting (29) and (30) to (21), we obtain

$$U_1 = A_1 + \widetilde{A_1}$$

$$= \int_{\mathbb{R}^n} (\rho^{-a}(\log\rho)^c)^2 \left|\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}\right|^2 dz$$

$$+\int_{\mathbb{R}^n}\overline{f(z)}\left(\frac{m+k|x|^{\gamma}}{\rho^{\gamma+1}} - \frac{(1+\gamma)|x|^{2\gamma+2}+|x|^{\gamma}(1+\gamma)^2|y|^2}{\rho^{3\gamma+3}}\right)\times\left((\rho^{-a}(\log\rho)^c)^2\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}\right)dz$$

$$+\int_{\mathbb{R}^n}\rho^{-a}(\log\rho)^c\overline{f(z)}\frac{z\cdot\nabla_\gamma\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}dz. \quad (31)$$

Summing (31), (15), and (17), we get

$$U_1 + U_3 + U_5 = \int_{\mathbb{R}^n}(\rho^{-a}(\log\rho)^c)^2\left|\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}\right|^2 dz. \quad (32)$$

Now we consider the sum of (14) and (19):

$$U_2 + U_7 = -\alpha\int_{\mathbb{R}^n}\overline{f(z)}\,\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma\rho^{-b}(log\,\rho)^d f\right)(z)}{\rho^{\gamma+1}}dz$$
$$+\alpha\int_{\mathbb{R}^n}\overline{f(z)}\,\rho^{-b}(\log\rho)^d\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}dz$$

$$= -\alpha\left(\int_{\mathbb{R}^n}\frac{\overline{f(z)}\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}\left(\sum_{i=1}^{m}x_i\,\partial_{x_i}\left(\rho^{-b}(\log\rho)^d f(z)\right) + |x|^{\gamma}\sum_{j=1}^{k}y_j\,\partial_{y_j}\left(\rho^{-b}(\log\rho)^d f(z)\right)\right)dz\right)$$

$$+\alpha\int_{\mathbb{R}^n}\overline{f(z)}\,\rho^{-b}(\log\rho)^d\rho^{-a}(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}dz$$

$$= -\alpha\int_{\mathbb{R}^n}|f(z)|^2\,\rho^{-a}(\log\rho)^c\frac{z\cdot\nabla_\gamma\rho^{-b}(\log\rho)^d}{\rho^{\gamma+1}}dz. \quad (33)$$

Thus, putting (32), (33), (16), (18), (20) to (12), we obtain the following:

$$0 \le \int_{\mathbb{R}^n}\left|\left(T_{\alpha,\gamma,a,b,c,d}f\right)(z)\right|^2 dz$$

$$= \int_{\mathbb{R}^n}(\rho^{-a}(\log\rho)^c)^2\left|\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}\right|^2 dz$$

$$-\alpha\int_{\mathbb{R}^n}\left(\rho^{-a}(\log\rho)^c\frac{z\cdot\nabla_\gamma\rho^{-b}(\log\rho)^d}{\rho^{\gamma+1}} + \rho^{-b}(\log\rho)^d\frac{z\cdot\nabla_\gamma\rho^{-a}(\log\rho)^c}{\rho^{\gamma+1}}\right)|f(z)|^2\,dz$$

$$-\alpha\int_{\mathbb{R}^n}\left(\frac{m+k|x|^{\gamma}}{\rho^{\gamma+1}}-\frac{(1+\gamma)|x|^{2\gamma+2}+|x|^{\gamma}(1+\gamma)^2|y|^2}{\rho^{3\gamma+3}}\right)$$

$$\times\,\rho^{-a}(\log\rho)^{c}\rho^{-b}(\log\rho)^{d}|f(z)|^2dz$$

$$+\alpha^2\int_{\mathbb{R}^n}(\rho^{-b}(\log\rho)^{d})^2|f(z)|^2\,dz$$

$$=L_1+L_2+L_3+L_4+L_5, \tag{34}$$

where

$$L_1=\int_{\mathbb{R}^n}\frac{(\log\rho)^{2c}}{\rho^{2a}}\left|\frac{z\cdot(\nabla_{\gamma}f)(z)}{\rho^{\gamma+1}}\right|^2dz,$$

$$L_2=-\alpha\int_{\mathbb{R}^n}\left(\rho^{-a}(\log\rho)^{c}\frac{z\cdot\nabla_{\gamma}(\rho^{-b}(\log\rho)^{d})}{\rho^{\gamma+1}}\right)|f(z)|^2\,dz,$$

$$L_3=-\alpha\int_{\mathbb{R}^n}\left(\rho^{-b}(\log\rho)^{d}\frac{z\cdot\nabla_{\gamma}(\rho^{-a}(\log\rho)^{c})}{\rho^{\gamma+1}}\right)|f(z)|^2\,dz,$$

$$L_4=-\alpha\int_{\mathbb{R}^n}\left(\frac{m+k|x|^{\gamma}}{\rho^{\gamma+1}}-\frac{(1+\gamma)|x|^{2\gamma+2}+|x|^{\gamma}(1+\gamma)^2|y|^2}{\rho^{3\gamma+3}}\right)\times\frac{(\log\rho)^{c+d}}{\rho^{a+b}}|f(z)|^2dz,$$

$$L_5=\alpha^2\int_{\mathbb{R}^n}\frac{(\log\rho)^{2d}}{\rho^{2b}}|f(z)|^2\,dz.$$

For L_1, we have

$$L_1=\int_{\mathbb{R}^n}\frac{(\log\rho)^{2c}}{\rho^{2a}}\left|\frac{z\cdot(\nabla_{\gamma}f)(z)}{\rho^{\gamma+1}}\right|^2dz. \tag{35}$$

For L_2:

$$L_2=-\alpha\int_{\mathbb{R}^n}\left(\rho^{-a}(\log\rho)^{c}\frac{z\cdot\nabla_{\gamma}(\rho^{-b}(\log\rho)^{d})}{\rho^{\gamma+1}}\right)|f(z)|^2\,dz$$

$$=-\alpha\int_{\mathbb{R}^n}\left[\frac{(\log\rho)^{c}}{\rho^{\gamma+1+a}}\left[\sum_{i=1}^{m}x_i\,\partial_{x_i}\left(\frac{(\log\rho)^{d}}{\rho^{b}}\right)+|x|^{\gamma}\sum_{j=1}^{k}y_j\,\partial_{y_j}\left(\frac{(\log\rho)^{d}}{\rho^{b}}\right)\right]\right]|f(z)|^2\,dz$$

$$=-\alpha\int_{\mathbb{R}^n}\frac{(\log\rho)^{c}}{\rho^{\gamma+1+a}}\left(\sum_{i=1}^{m}x_i\,\partial_{x_i}\left(\frac{(\log\rho)^{d}}{\rho^{b}}\right)\right)|f(z)|^2\,dz$$

$$-\alpha\int_{\mathbb{R}^n}\frac{(\log\rho)^{c}}{\rho^{\gamma+1+a}}\left(|x|^{\gamma}\sum_{j=1}^{k}y_j\,\partial_{y_j}\left(\frac{(\log\rho)^{d}}{\rho^{b}}\right)\right)|f(z)|^2\,dz$$

$$=-\alpha\int_{\mathbb{R}^n}\frac{(\log\rho)^{c}}{\rho^{\gamma+1+a}}\left(\frac{(\log\rho)^{d}|x|^{2\gamma+2}}{\rho^{b+2\gamma+2}}\left(\frac{d}{\log\rho}-b\right)\right)|f(z)|^2\,dz$$

$$-\alpha\int_{\mathbb{R}^n}\frac{(\log\rho)^c}{\rho^{\gamma+1+a}}\left(\frac{(\log\rho)^d(1+\gamma)|x|^\gamma|y|^2}{\rho^{b+2\gamma+2}}\left(\frac{d}{\log\rho}-b\right)\right)|f(z)|^2\,dz$$

$$=-\alpha\int_{\mathbb{R}^n}\frac{(\log\rho)^{c+d}(|x|^{2\gamma+2}+|x|^\gamma(1+\gamma)|y|^2)}{\rho^{a+b+3\gamma+3}}\left(\frac{d}{\log\rho}-b\right)|f(z)|^2\,dz. \quad (36)$$

By interchanging a and b as well as d and c in (35), we can immediately get L_3:

$$L_3=-\alpha\int_{\mathbb{R}^n}\left(\rho^{-b}(\log\rho)^d\frac{z\cdot\nabla_\gamma(\rho^{-a}(\log\rho)^c)}{\rho^{\gamma+1}}\right)|f(z)|^2\,dz$$

$$=-\alpha\int_{\mathbb{R}^n}\frac{(\log\rho)^{c+d}(|x|^{2\gamma+2}+|x|^\gamma(1+\gamma)|y|^2)}{\rho^{a+b+3\gamma+3}}\left(\frac{c}{\log\rho}-a\right)|f(z)|^2\,dz. \quad (37)$$

Now for L_4, we get

$$L_4=-\alpha\int_{\mathbb{R}^n}\left(\frac{m+k|x|^\gamma}{\rho^{\gamma+1}}\right)\frac{(\log\rho)^{c+d}}{\rho^{a+b}}|f(z)|^2dz$$

$$+\alpha\int_{\mathbb{R}^n}\left(\frac{(1+\gamma)|x|^{2\gamma+2}+|x|^\gamma(1+\gamma)^2|y|^2}{\rho^{3\gamma+3}}\right)\frac{(\log\rho)^{c+d}}{\rho^{a+b}}|f(z)|^2\,dz. \quad (38)$$

Finally, for L_5 we have

$$L_5=\alpha^2\int_{\mathbb{R}^n}\frac{(\log\rho)^{2d}}{\rho^{2b}}|f(z)|^2\,dz. \quad (39)$$

Putting (35)-(39) to (34), we get

$$0\le\int_{\mathbb{R}^n}\left|\left(T_{\alpha,\gamma,a,b,c,d}f\right)(z)\right|^2dz=\int_{\mathbb{R}^n}(\rho^{-a}(\log\rho)^c)^2\left|\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{\gamma+1}}\right|^2dz$$

$$+\alpha\int_{\mathbb{R}^n}\left[\frac{(\log\rho)^{c+d}(|x|^{2\gamma+2}+|x|^\gamma(1+\gamma)|y|^2)}{\rho^{a+b+3\gamma+3}}\left(a+b+1+\gamma-\frac{c+d}{\log\rho}\right)\right]|f(z)|^2\,dz$$

$$+\alpha\int_{\mathbb{R}^n}\left[-\frac{m+k|x|^\gamma}{\rho^{a+b+1+\gamma}}(\log\rho)^{c+d}+\alpha\frac{(\log\rho)^{2d}}{\rho^{2b}}\right]|f(z)|^2\,dz. \quad (40)$$

Meanwhile, using (6), we have

$$\left\|\left(T_{\alpha,\gamma,a,b,c,d}f\right)(z)\right\|^2_{L^2(\mathbb{R}^n)}=\left\|(\log\rho)^c\frac{z\cdot\left(\nabla_\gamma f\right)(z)}{\rho^{a+\gamma+1}}+\alpha\frac{(\log\rho)^d}{\rho^b}f(z)\right\|^2_{L^2(\mathbb{R}^n)}. \quad (41)$$

Putting (40) and (41) together, we obtain (3).

Conclusion

In conclusion, we have successfully derived a weighted Hardy identity associated with the Baouendi-Grushin operator using the method of factorization. In addition, we were able to apply our result in the magnetic inequality related to the Landau-Hamiltonian type magnetic field. This approach allowed us to extend classical Hardy inequalities to more complex settings involving degenerate elliptic operators.

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БАОУЭНДИ-ГРУШИН ВЕКТОРЛЫҚ ӨРІСТЕРІНЕ ЖӘНЕ ЛАНДАУ-ГАМИЛЬТОНИАНҒА БАЙЛАНЫСТЫ ХАРДИ ТЕҢСІЗДІКТЕРІ МЕН ТЕҢДІКТЕРІ

Аңдатпа

Бұл мақалада біз Баоуэнди-Грушин векторлық өрістерімен байланысты салмақты Харди теңдігін талқылай отырып және оның дифференциалдық теңсіздіктер контекстінде әртүрлі қолданыстарын зерттейміз. Бұл Харди теңдігінен, сәйкес параметрлерді таңдау арқылы Баоуэнди-Грушин операторына қатысты Харди типті теңсіздіктер үшін нақты қалдық формулаларын да алуға болады. Коммутативті жағдайды қарастырғанда, бізге евклидтік кеңістік үшін де, Баоуэнди-Грушин операторына қатысты жақсартылған салмақты Харди теңсіздіктерін де анықтау мүмкіндігі туады. Мысалы, нақты жағдайда, Баоуэнди-Грушин операторына қатысты теріс емес қалдық мүшелерді алып тастап және тиісті параметрлерді дәл таңдау арқылы, біздің теңдіктен топологиялық өлшемге тәуелді емес тұрақтысы дәл анықталған радиалды туынды операторына арналған жақсартылған критикалық Харди теңсіздігін алуға да мүмкіндік береді. Сонымен қатар, осы мақалада Гештези мен Литтлджон [1] жұмысында ұсынылған дифференциалды өрнектерді факторизациялау әдісін де қолданамыз. Бұл мақалада біз коммутативті емес Баоуэнди-Грушин вектордық өрісінде факторизация әдісін қолдануды көрсетеміз. Сонымен бірге, Баоуэнди-Грушин векторлық өрістерімен байланысты Харди теңдігінің қолданысы ретінде, жалпыланған Ландау-Гамильтон (немесе бұралған лапласиан) операторы үшін қалдық мүшелері бар Харди типті теңсіздікті аламыз.

Тірек сөздер: факторизация әдісі, Харди теңсіздігі, Баоуэнди-Грушин операторы, Ландау-Гамильтониан.

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НЕРАВЕНСТВА И ТОЖДЕСТВА ХАРДИ, СВЯЗАННЫЕ С ВЕКТОРНЫМИ ПОЛЯМИ БАОУЭНДИ-ГРУШИНА И ЛАНДАУ-ГАМИЛЬТОНИАНОМ

Аннотация

В этой статье мы представляем взвешенное тождество Харди, связанное с векторными полями Баоуэнди-Грушина, и его приложения с применениями в контексте дифференциальных неравенств. С помощью выбора соответствующих параметров наше полученное тождество Харди, связанное с оператором Баоуэнди-Грушина, влечет за собой многочисленные формулы точного остатка для неравенств типа Харди. В коммутативном случае мы получаем улучшенные взвешенные неравенства Харди в постановке евклидова пространства. Например, в частном случае, отбрасывая неотрицательные остаточные члены, связанные с оператором Баоуэнди-Грушина, и выбирая подходящие параметры, наше тождество позволяет нам вывести улучшенное критическое неравенство Харди для радиального производного оператора с точной константой, которая, в свою очередь, не зависит от топологической размерности. Мы используем метод факторизации дифференциальных выражений, использованный Гештези и Литтлджоном в [1]. В данной статье мы демонстрируем применение метода факторизации в некоммутативной постановке Баоуэнди-Грушина. В качестве применения полученного тождества Харди, связанного с векторными полями Баоуэнди-Грушина, мы устанавливаем неравенство Харди для обобщенного гамильтониана Ландау (или искривленного лапласиана) с определенными остаточными членами.

Ключевые слова: метод факторизации, неравенство Харди, оператор Баоуэнди-Грушина, Ландау-Гамильтониан.

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