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ON A SPECTRAL PROBLEM FOR THE LAPLACE OPERATOR WITH MORE GENERAL BOUNDARY CONDITIONS

Abstract

In this paper, we consider a spectral problem for the Laplace operator with more general boundary conditions in a unit disk B_1. In the special cases, the boundary conditions inlude periodic and Samarskii-Ionkin type boundary conditions. The main important property of our problem is its non-self-adjointness, which causes number of difficulties in their analytical and numerical solutions. For example, the Fourier method of separation of variables cannot be applied directly to our problem. Therefore, the possibility of separation of variables is justified in this paper. Namely, we present a method that reduces solution of the problem to a sequential solution of two classical local boundary value problems. By using this method, we construct all eigenfunctions and eigenvalues of the problem in explicit forms. Moreover, completeness of the system of the eigenfunctions is proved in $L^2(B_1)$. Notably, our result generalises the special case of the result on the two-dimensional periodic boundary value problem for the Laplace operator obtained in [1–2].

Key words: Laplace operator, Samarskii-Ionkin type problem, eigenfunctions, eigenvalues.

Introduction

In [1], as a two-dimensional analogue to the classical periodic boundary value problems the authors considered the Poisson equation

$$-\Delta u = f(z),\ z \in B_1 \tag{1.1}$$

in a unit disk $B_1 = \{z = (x, y) = x + iy \in C: |z| < 1\}$ with the periodic boundary conditions

$$u(1,\varphi) - (-1)^k u(1,\varphi + \pi) = \tau(\varphi), \qquad 0 \le \varphi \le \pi, \tag{1.2}$$

$$\frac{\partial u}{\partial r}(1,\varphi) + (-1)^k \frac{\partial u}{\partial r}(1,\varphi + \pi) = \nu(\varphi),\ \ 0 \le \varphi \le \pi, \tag{1.3}$$

where $f(z) \in C^\gamma\left(\overline{B_1}\right)$, $\tau(\varphi) \in C^{1+\gamma}[0,\pi]$, and $\nu(\varphi) \in C[0,\pi]$, $0 < \gamma < 1, k = 1,2$. In [1], the self-adjointness of these problems was shown, and all corresponding eigenvalues and eigenfunctions were constructed. The problem (1.1)-(1.3) is referred to as antiperiodic when $k = 1$ and periodic when $k = 2$.

In [2] the authors considered the Poisson equation (1.1) with the following boundary conditions

$$u(1,\varphi) + u(1,2\pi - \varphi) = \tau(\varphi), \qquad 0 \le \varphi \le \pi \tag{1.4}$$

$$\frac{\partial u}{\partial r}(1,\varphi) - \frac{\partial u}{\partial r}(1,2\pi - \varphi) = \nu(\varphi),\ \ 0 \le \varphi \le \pi, \tag{1.5}$$

where $f(z) \in C^{\gamma}(\overline{B_1})$, $\tau(\varphi) \in C^{1+\gamma}[0,\pi]$, and $\nu(\varphi) \in C[0,\pi]$, $0 < \gamma < 1$. The eigenvalues of the spectral problem (1.1), (1.4)-(1.5), with $\tau = \nu = 0$, consist of all the eigenvalues of the Dirichlet and Neumann problems, and each eigenvalue corresponds to one eigenfunction. As for the spectral problem (1.1)-(1.3) with $k = 1$ and $\tau = \nu = 0$, its eigenvalues consist of only a part ("half") of the eigenvalues $\mu_k^{(n)}$ of the Dirichlet problem for $n = 2j$ and a part ("half") of the eigenvalues $\mu_k^{(n)}$ of the Neumann problem for $n = 2j + 1$. Additionally, it was noted that each eigenvalue has two corresponding eigenfunctions. For the multidimensional extension of these problems, we refer the reader to [2].

Later, in [3] spectral problems for the Laplace operator with the following more general boundary conditions

$$u(1,\varphi) - \alpha u(1,2\pi-\varphi) = 0, \qquad 0 \le \varphi \le \pi, \qquad \alpha \ne 1 \qquad (1.6)$$

$$\frac{\partial u}{\partial r}(1,\varphi) - \frac{\partial u}{\partial r}(1,2\pi-\varphi) = 0, \quad 0 \le \varphi \le \pi \qquad (1.7)$$

were investigated. One can note that this problem reduces the antiperiodic problem when $\alpha = -1$ from [2] and Samarksii-Ionkin type problem from [4] when $\alpha = 0$. Observe that the latter problem is non-self-adjoint when $\alpha \ne -1$. Nevertheless, the author in [3] managed to prove completeness of its eigenfunctions. For studies on the well-posedness of the Poisson equation with inhomogeneous conditions, we refer to [4] when $\alpha = 0$ as well as to [5] and [6] with general α in two and multidimensional cases, respectively. For other generalisations of (1.1)-(1.3) we can refer to e.g. [7] and [8]. We can also refer to [9] and [10] for the Samarskii-Ionkin type non-local problems concerning other partial differential equations.

Here, in this paper we consider similar problem with a parameter in the second boundary condition (1.7): Let $B_1 = \{z = (x,y) = x + iy \in C: |z| < 1\}$ be a unit disk, $r = |z|, \varphi = \arctan(y/x)$, $B_1^+ = B_1 \cap \{y > 0\}$, and $B_1^- = B_1 \cap \{y < 0\}$. We consider the spectral problem corresponding to the Laplace operator

$$-\Delta u(z) = \lambda u(z), \ |z| < 1 \qquad (1.8)$$

with the boundary conditions

In the special case $\beta = -1$ the problem (1.8)-(1.10) becomes the periodic boundary value problem from [2].

Note that the problem (1.8)-(1.10) is non-self-adjoint in general, so the direct use of the method

$$u(1,\varphi) - u(1,2\pi-\varphi) = 0, \ 0 \le \varphi \le \pi \qquad (1.9)$$

$$\frac{\partial u}{\partial r}(1,\varphi) - \beta\frac{\partial u}{\partial r}(1,2\pi-\varphi) = 0, \ 0 \le \varphi \le \pi, \qquad \beta \in \mathbb{R}. \qquad (1.10)$$

of separation of variables is impossible. Here, in this paper we propose another method that reduces the solution of the problem to a sequential solution of two classical local boundary value problems. By this method, we calculate eigenfunctions and eigenvalues of the problems (1.8)-(1.10) in an explicit form. Furthermore, we prove completeness of the eigenfunctions.

Material and methods

As mentioned in Introduction, as analogues of the classical periodic and anti-periodic problems the authors in [1, 2] considered two and multidimensional versions for the Laplace operator. Then, in [3–6] these problems but with more generalised Samarskii-Ionkin type boundary conditions that include periodic, antiperiodic and Samarskii-Ionkin type boundary conditions were investigated. We also refer to [7–10] for other generalisations. Here, we are interested in a non-self-adjoint spectral problem, for which it is impossible to directly apply traditional method of separation of variables. In this paper, we will demonstrate a method that reduces the solution of the problem to a sequential solution of two classical local boundary value problems.

Results and Discussion

In this section, we present our main result. Before doing so, we will introduce the necessary notations.

We will use L_β to denote the closure in $L^2(B_1)$ of the operator defined by the differential expression $\ell_1 u = -\Delta u(z)$ on the linear manifold of functions $u(z) \in C^{2+\gamma}(B_1)$, $0 < \gamma < 1$, satisfying the following two boundary conditions

$$u(1,\varphi) - u(1,2\pi-\varphi) = 0,\ \frac{\partial u}{\partial r}(1,\varphi) - \beta\frac{\partial u}{\partial r}(1,2\pi-\varphi) = 0,\ 0 \le \varphi \le \pi, \beta \in \mathbb{R}.$$

Now we are ready to state our result on the spectral problem L_β :

Theorem 1. Let $\beta \neq 1$. Then the system of the eigenfunctions of the operator L_β takes the form

$$u_k^1(z) = J_k\left(r\sqrt{\lambda_N}\right)\cos k\varphi\,, \qquad 0 \le \varphi \le 2\pi, \qquad k = 0,1,2,\ldots \tag{2.1}$$

$$u_m^2(z) = J_m\left(r\sqrt{\lambda_D}\right)\sin m\varphi + \frac{a_0}{2}J_0\left(r\sqrt{\lambda_D}\right) + \sum_{n=1,n\neq m}^{\infty} a_n J_n\left(r\sqrt{\lambda_D}\right)\cos n\varphi \tag{2.2}$$

for all $0 \le \varphi \le 2\pi, m = 1,2,\ldots$, where

$$a_n = -\frac{2\sqrt{\lambda_D}J_m'\left(\sqrt{\lambda_D}\right)(1+\beta)}{\pi J_n\left(\sqrt{\lambda_D}\right)(1-\beta)}\int_0^\pi \sin m\psi \cos n\psi d\psi, n \neq m, n = 0,1,\ldots$$

Here, $J_i(x), i = 0,1,\ldots$ are Bessel functions, λ_D and λ_N are eigenvalues of the Dirichlet and Neumann problems for the Laplace equation in B_1, respectively. Moreover, the system of the eigenfunctions of the operator L_β is complete in $L^2(B_1)$.

Proof of Theorem 1. Let us begin by introducing the auxiliary functions

$$c(r,\varphi) = \tfrac{1}{2}\left(u(r,\varphi) + u(r,2\pi-\varphi)\right),\ s(r,\varphi) = \tfrac{1}{2}\left(u(r,\varphi) - u(r,2\pi-\varphi)\right) \tag{2.3}$$

A direct calculation shows that the functions $c(z)$ and $s(z)$ satisfies the following spectral problems: for the function $s(z)$, we have the Dirichlet problem

$$-\Delta s(z) = \lambda s(z),\ \ z \in B_1;\ s(1,\varphi) = 0,\ \ 0 \le \varphi \le 2\pi \tag{2.4}$$

and for the function $c(z)$, we obtain the Neumann problem

$$-\Delta c(z) = \lambda c(z),\ \ z \in B_1;\quad \frac{\partial c}{\partial r}(1,\varphi) = \begin{cases} -\frac{1+\beta}{1-\beta}\frac{\partial s}{\partial r}(1,\varphi), & 0 \le \varphi \le \pi \\ \frac{1+\beta}{1-\beta}\frac{\partial s}{\partial r}(1,\varphi), & \pi \le \varphi \le 2\pi \end{cases} \tag{2.5}$$

Further, we split the rest of the proof into two cases:

In the case of $\lambda \neq \lambda_D$, it can be observed that $s(r,\varphi) = 0$, and the Neumann problem (2.5) becomes

$$-\Delta c(z) = \lambda c(z),\ \ z \in B_1;\ \frac{\partial c}{\partial r}(1,\varphi) = 0,\ 0 \le \varphi \le 2\pi. \tag{2.6}$$

Due to the property $c(r,\varphi) = c(r,2\pi-\varphi)$ from (2.3), one of the series of the eigenfunctions of the spectral problem L_β has the form

$$u_k(z) = J_k\left(r\sqrt{\lambda_N}\right)\cos k\varphi\,,\ \ k = 0,1,\ldots \tag{2.7}$$

It remains to consider the case $\lambda = \lambda_D$. In this scenario, by using the property $s(r,\varphi) = -s(r, 2\pi - \varphi)$ from the representation (2.3), we get

$$s_m(z) = J_m\left(r\sqrt{\lambda_D}\right)\sin m\varphi\,,\ m = 1,2,\ldots \tag{2.8}$$

Substituting this representation into the Neumann problem (2.5) implies

Recalling $c(r,\varphi) = c(r, 2\pi - \varphi)$, we seek the function $c(r,\varphi)$ in the form

$$c(r,\varphi) = \frac{a_0}{2} J_0\left(r\sqrt{\lambda_D}\right) + \sum_{n=1}^{\infty} a_n J_n\left(r\sqrt{\lambda_D}\right)\cos n\varphi \tag{2.11}$$

Plugging this into the boundary condition (2.10), we calculate

$$\begin{aligned} a_n J_n\left(\sqrt{\lambda_D}\right) &= -\int_0^{\pi} \frac{\sqrt{\lambda_D}(1+\beta)}{\pi(1-\beta)} J_m'\left(\sqrt{\lambda_D}\right)\sin m\psi \cos n\psi d\psi \\ &+ \int_{\pi}^{2\pi} \frac{\sqrt{\lambda_D}(1+\beta)}{\pi(1-\beta)} J_m'\left(\sqrt{\lambda_D}\right)\sin m\psi \cos n\psi d\psi \\ &= -\int_0^{\pi} \frac{2\sqrt{\lambda_D}(1+\beta)}{\pi(1-\beta)} J_m'\left(\sqrt{\lambda_D}\right)\sin m\psi \cos n\psi d\psi\,, \qquad n = 0,1,\ldots \end{aligned}$$

that is,

$$-\Delta c(z) = \lambda_D c(z), \qquad z \in B_1, \tag{2.9}$$

$$\frac{\partial c}{\partial r}(1,\varphi) = \begin{cases} -\dfrac{1+\beta}{1-\beta}\sqrt{\lambda_D} J_m'\left(\sqrt{\lambda_D}\right)\sin m\varphi\,, & 0 \le \varphi \le \pi; \\ \dfrac{1+\beta}{1-\beta}\sqrt{\lambda_D} J_m'\left(\sqrt{\lambda_D}\right)\sin m\varphi\,, & \pi \le \varphi \le 2\pi. \end{cases} \tag{2.10}$$

$$a_n = -\frac{2\sqrt{\lambda_D} J_m'\left(\sqrt{\lambda_D}\right)}{\pi J_n\left(\sqrt{\lambda_D}\right)} \int_0^{\pi} \frac{1+\beta}{1-\beta} \sin m\psi \cos n\psi d\psi\,, \qquad n = 0,1,\ldots$$

for $n \neq m$ and $a_n = 0$ for $n = m$.

Thus, we have completed the construction of the eigenvalues of the L_β problem:

$$u_k^1(z) = J_k\left(r\sqrt{\lambda_N}\right)\cos k\varphi\,, \qquad 0 \le \varphi \le 2\pi, k = 0,1,2,\ldots \tag{2.12}$$

$$u_m^2(z) = J_m\left(r\sqrt{\lambda_D}\right)\sin m\varphi + \frac{a_0}{2} J_0\left(r\sqrt{\lambda_D}\right) + \sum_{n=1, n\neq m}^{\infty} a_n J_n\left(r\sqrt{\lambda_D}\right)\cos n\varphi \tag{2.13}$$

for all $0 \le r \le 1,\ 0 \le \varphi \le 2\pi,\ m = 1,2,\ldots$.

Here, the convergence of the obtained series in (2.13) can be verified by using asymptotic forms of the Bessel function and Leibniz criterion.

Now, we need to show the second part of our result, namely that the obtained eigenfunctions (2.12) and (2.13) are complete in $L^2(B_1)$. For this, we note that

$$\int_{B_1} u_k^1(z) f(z) dz = \int_0^1 \int_0^{\pi} r J_k\left(r\sqrt{\lambda_N}\right)\left(f(r,\varphi) + f(r, 2\pi - \varphi)\right)\cos k\varphi dr d\varphi = 0$$

Using the fact that the system $\{rJ_k(r\sqrt{\lambda_N})\cos k\varphi\}_{k=0}^{k=\infty}$ is complete in $L^2(B_1^+)$, we derive from above that

$$f(r,\varphi)+f(r,2\pi-\varphi)=0, 0\le\varphi\le\pi. \quad (2.14)$$

Taking into account this, we obtain

$$\int_{B_1} u_m^2(z)f(z)dz=\int_{B_1}\left(J_m(r\sqrt{\lambda_D})\sin m\varphi\right)f(z)dz$$
$$+\int_{B_1}\left(\frac{a_0}{2}J_0(r\sqrt{\lambda_D})+\sum_{n=1,n\neq m}^{\infty} a_nJ_n(r\sqrt{\lambda_D})\cos n\varphi\right)f(z)dz$$
$$=\int_0^1\int_0^\pi rJ_m(r\sqrt{\lambda_D})\sin m\varphi(f(r,\varphi)-f(r,2\pi-\varphi))drd\varphi=0.$$

Here, using the completeness of $\{rJ_m(r\sqrt{\lambda_D})\sin m\varphi\}_{m=1}^{m=\infty}$ in $L^2(B_1^+)$, we conclude that

$$f(r,\varphi)-f(r,2\pi-\varphi)=0, \quad 0\le\varphi\le\pi. \quad (2.15)$$

Thus, a combination of (2.14) and (2.15) implies $f(r,\varphi)=0$ for $0\le\varphi\le2\pi$, which yields the completeness of the eigenfunctions (2.12) and (2.13) in $L^2(B_1)$, as desired.

Conclusion

Thus, we showed a method for the non-self-adjoint spectral problem that reduces the solution of the problem to a sequential solution of two classical local boundary value problems. Namely, this method allowed us to construct eigenfunctions and eigenvalues in explicit form. Moreover, we proved completeness of the eigenfunctions in $L^2(B_1)$. It should be noted that the question of whether the system of eigenfunctions we have constructed forms an unconditional basis in $L^2(B_1)$ remains open.

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ЖАЛПЫЛАНҒАН ШЕКТІК ШАРТТАРЫ БАР ЛАПЛАС ОПЕРАТОРЫ ҮШІН ҚОЙЫЛҒАН СПЕКТРАЛДЫҚ ЕСЕП ТУРАЛЫ

Аңдатпа

Бұл мақалада B_1 бірлік шеңберінде Лаплас операторы үшін жалпы шекаралық шарттары бар спектрлік есеп қарастырылады. Дербес жағдайларда шекаралық шарттар периодтық және Самарский-Ионкин типіндегі шекаралық шарттарды қамтиды. Есептің маңызды ерекшелігі – оның өзіне-өзі түйіндес еместігі, бұл аналитикалық және сандық шешуде бірқатар қиындықтар туғызады. Мысалы, Фурье айнымалыларды бөлу әдісі тікелей қолданылмайды. Осыған байланысты жұмыста айнымалыларды бөлу әдісінің қолдану мүмкіндігі негізделеді. Атап айтқанда, есепті екі локальды классикалық шекаралық есептерді тізбектей шешуге келтіретін әдіс ұсынылады. Бұл әдісті пайдалана отырып, есептің барлық меншікті функциялары мен меншікті мәндері анық түрде анықталды. Сонымен қатар, меншікті функциялар жүйесінің кеңістігінде толықтығы дәлелденді. Айта кету керек, бұл нәтиже Лаплас операторы үшін екі өлшемді периодтық шекаралық есептің [1, 2] еңбектерінде алынған нәтижелерінің арнайы жағдайын жалпылайды.

Тірек сөздер: Пуассон теңдеуі, Самарск-Ионкин типті есеп, меншікті функциялар, меншікті мәндер.

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ОБ ОДНОЙ СПЕКТРАЛЬНОЙ ЗАДАЧЕ ДЛЯ ОПЕРАТОРА ЛАПЛАСА С БОЛЕЕ ОБЩИМИ ГРАНИЧНЫМИ УСЛОВИЯМИ

Аннотация

В данной работе мы рассматриваем спектральную задачу для оператора Лапласа с более общими краевыми условиями в единичном круге B_1. В частных случаях краевые условия включают периодические и краевые условия типа Самарского-Ионкина. Основное важное свойство нашей задачи – это ее несамосопряженность, что вызывает ряд трудностей при аналитических и численных решениях. Например, метод Фурье для разделения переменных не может быть применен напрямую к нашей задаче. Поэтому в данной работе обосновывается возможность применения метода разделения переменных. А именно мы представляем метод, который сводит решение задачи к последовательному решению двух классических локальных краевых задач. С использованием этого метода мы строим все собственные функции и собственные значения задачи в явном виде. Более того, доказывается полнота системы собственных функций в $L^2(B_1)$. Примечательно, что наш результат обобщает частный случай решения двумерной задачи с периодическими краевыми условиями для оператора Лапласа, полученного в [1, 2].

Ключевые слова: уравнение Пуассона, задача типа Самарского–Ионкина, собственные функции, собственные значения.

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