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ANALYTICAL SOLUTIONS OF THE NONLOCAL NONLINEAR SCHRÖDINGER-TYPE EQUATIONS

Abstract

In physics, nonlinear equations are applied to characterize the varied phenomena. Usually, the nonlinear equations are presented by nonlinear partial differential equations, that can be received as conditions for the compatibility of two linear differential equations, named the Lax pairs. The presence of the Lax pair determines integrability for the nonlinear partial differential equation. Linked to this development was the realization that certain coherent structures, known as solitons, which play a fundamental role in nonlinear phenomena as lattice dynamics, nonlinear optics, and fluid mechanics. One of the famous equations is the nonlinear Schrödinger equation which is associated with various physical phenomena in nonlinear optics and Bose-Einstein condensates. This equation allows the Lax pair thus it is integrable. This work investigates nonlocal nonlinear Schrödinger-type equations with PT symmetry. Nonlocal nonlinear equations arise in various physical contexts as fluid dynamics, condensed matter physics, optics, and so on. We introduce the Lax pair formulation for the nonlocal nonlinear Schrödinger-type equations. The method of the Darboux transformation is applied to receive analytical solutions.

Key words: Schrodinger-type equations, Darboux transformation, nonlocal, Lax pair, analytical solution, symmetry.

Introduction

The nonlinear partial differential equations (NPDE) are studied in several works [1–2]. One of the famous equations is the nonlinear Schrödinger equation (NLSE). This equation is a universal model, that describes the evolution of quasi-monochromatic and weakly nonlinear wave trains in media with cubic nonlinearities. In optics, the NLSE is the main model that characterizes the propagation of optical waves in Kerr media [3]. Various generalizations and modifications for the NLSE are studied. For example, in [4] for the unstable NLSE the exact solutions are found by the improved modified Sardar sub-equation method. The derivative nonlinear Schrödinger hierarchy is studied in [5]. In work [6] dynamics of optical solitons in the fifth-order nonlinear Schrödinger equation is investigated.

The most part of NPDEs is local. This denotes that the local solution evolution relies only on the local solution value and its local space and time derivatives. Latterly, the one-dimensional nonlocal equations are studied in [7–9]. The primary one-dimensional nonlocal equation was the PT-symmetric nonlinear Schrödinger equation (NLSE) [10–12]. At present time there are different methods to explore the NPDE, such as the method of the Darboux transformation (DT) [13–18], the extended tanh method [19, 20], the bilinear Hirota method [21–23], the sine-cosine [24, 25], and others.

In this research, the nonlocal nonlinear Schrödinger-type equations are studied

$$iq_t(x,y,t) + q_{xy}(x,y,t) - v(x,y,t)q(x,y,t) + \alpha q(x,y,t) - i\beta q_x(x,y,t) = 0, \quad (1)$$

$$v_x(x,y,t) + 2(q(x,y,t)q^*(-x,-y,t))_y = 0, \quad (2)$$

where q, v are functions, α, β are constants, * means a complex conjugate. Eqs. (1)–(2) suppose the following reductions:

If $\alpha = 0, \beta = 0$, and local case $r = q^*(x,y,t)$ we get the two-dimensional NLSE:

$$iq_t(x,y,t) + q_{xy}(x,y,t) - v(x,y,t)q(x,y,t) = 0, \quad (3)$$

$$v_x(x,y,t) + 2(q(x,y,t)q^*(x,y,t))_y = 0. \quad (4)$$

Eqs. (3)–(4) are studied in [26] by the sine-cosine method where the researcher found the traveling wave solutions. The conservation laws are obtained in [27].

Classical (local) case $r = q^*(x,y,t)$ of Eqs. (1)-(2) has the following form

$$iq_t(x,y,t) + q_{xy}(x,y,t) - v(x,y,t)q(x,y,t) + \alpha q(x,y,t) - i\beta q_x(x,y,t) = 0, \quad (5)$$

$$v_x(x,y,t) + 2(q(x,y,t)q^*(x,y,t))_y = 0. \quad (6)$$

Various solutions of Eqs. (5)–(6) are obtained in [28] by the bilinear Hirota method and the extended tanh method. However, nonlocal forms for Eqs (5)–(6) have not been found in other studies, which will be the main focus of this paper.

In this work by the Lax pair, we introduce the nonlinear Schrödinger-type equations in nonlocal form with AKNS reduction $r = q^*(-x,-y,t)$. Eqs. (1)–(2) are called PT- symmetric because is invariant under the action if the PT operator, i.e. the joint transformation $x \to -x, y \to -y$. As a method to research we use the Darboux transformation which is an effective tool to solve nonlinear partial differential equations and receive exact solutions for the equations (1)–(2).

Material and methods

In this part, we introduce the Lax pair and the DT for the nonlocal nonlinear Schrödinger-type equations.

Lax pair

The integrability of the nonlinear equation is provided by the Lax pair. Here, we introduce the Lax pair for (1)–(2) which can be presented by the next form

$$\Psi_x = U\Psi, \quad (7)$$

$$\Psi_t = 2\lambda\Psi_y + V\Psi, \quad (8)$$

where $\Psi = (\psi_1(x,y,t), \psi_2(x,y,t))^T$ (T define the transpose of the matrix), λ is const and the matrices V and U have the next expression

$$U = -i\lambda\sigma_3 + U_0, \tag{9}$$

$$V = \lambda V_1 + V_0, \tag{10}$$

with

$$V_0 = \begin{pmatrix} -\frac{iv}{2} + \frac{i\alpha}{2} & iq_y + \beta q \\ iq^*{}_y(-x,-y,t) - \beta q_y^*(-x,-y,t) & \frac{iv}{2} - \frac{i\alpha}{2} \end{pmatrix}, V_1 = \begin{pmatrix} -i\beta & 0 \\ 0 & i\beta \end{pmatrix}.$$

$$U_0 = \begin{pmatrix} 0 & q \\ -q^*(-x,-y,t) & 0 \end{pmatrix}. \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Over some calculations by (9)-(10), it can be verified that the zero-curvature condition:

$$U_t - V_x - 2\lambda U_y + UV - VU = 0,$$

directly gives rise to the equations (1)-(2).

Darboux transformation

The following transformation of the equations (1)–(2) based on the DT for the Ablowitz-Kaup-Newell-Segur (AKNS) system

$$\Psi^{[1]} = T\Psi = (\lambda I - P)\Psi,$$

where $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The new function $\Psi^{[1]}$ satisfies

$$\Psi_x^{[1]} = U^{[1]}\Psi^{[1]},$$
$$\Psi_t^{[1]} = 2\lambda\Psi_y^{[1]} + V^{[1]}\Psi^{[1]},$$

where $V^{[1]}$, $U^{[1]}$ rely on $q^{[1]}, v^{[1]}$ and λ. The connection between $q^{[1]}, v^{[1]}$ and $U^{[1]}$-$V^{[1]}$ is similar to the connection between q, v and U -V. Then, it is clear that the Darboux matrix T responds to the equations

$$T_x + TU = U^{[1]}T, \tag{11}$$

$$T_t + TV = 2\lambda T_y + V^{[1]}T. \tag{12}$$

Collecting the different powers of λ^i of the equations (11)–(12) and making some algebraic calculation on (11)–(12), we get a relation among functions $q^{[1]}$ and q:

$$q^{[1]}(x,y,t) = q(x,y,t) - 2ip_{12}, \tag{13}$$
$$q^{*[1]}(-x,-y,t) = q^*(-x,-y,t) - 2ip_{21},$$
$$v^{[1]} = v + 4ip_{11y} = v - 4ip_{22y}. \tag{14}$$

with a constraint $p_{12} = -p_{21}^*(-x,-y,t)$.

Main result

The one-fold Darboux transformation

In this section, we present one-fold Darboux transformation and obtain exact solutions.

Theorem. The one-fold Darboux transformation of the nonlocal nonlinear Schrödinger-type equations (1)-(2) is

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \tag{15}$$

where

$$p_{11} = \frac{1}{\Delta} * (\lambda_1 \psi_1(x,y,t)\psi_1^*(-x,-y,t) - \lambda_1^* \psi_2(x,y,t)\psi_2^*(-x,-y,t)),$$

$$p_{12} = \frac{1}{\Delta} * ((\lambda_1 + \lambda_1^*)\psi_1(x,y,t)\psi_2^*(-x,-y,t)),$$

$$p_{21} = \frac{1}{\Delta} * ((\lambda_1 - \lambda_1^*)\psi_2(x,y,t)\psi_1^*(-x,-y,t)),$$

$$p_{22} = \frac{1}{\Delta} * (\lambda_1 \psi_2(x,y,t)\psi_2^*(-x,-y,t) - \lambda_1^* \psi_1(x,y,t)\psi_1^*(-x,-y,t)),$$

with $\Delta = \psi_1 \psi_1^*(-x,-y,t) - \psi_2 \psi_2^*(-x,-y,t)$.

Proof.

We now assume that

$$P = H \Lambda H^{-1}, \tag{16}$$

with

$$H = \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}, \ \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{17}$$

where $(f_1, f_2)^T = (\psi_1(x,y,t), \psi_2(x,y,t))^T$ is a solution to Eqs. (7)–(8) by $\lambda = \lambda_1$ and $(g_1, g_2)^T = (\psi_2^*(-x,-y,t), \psi_1^*(-x,-y,t))^T$ is the solution in the case $\lambda = -\lambda_1^* = \lambda_2$.

After substitution (17) into (16) and making some calculations, we can obtain the explicit form for the expression for P (15). This completes the proof. □

Therefore, the new solutions (13) and (14) can be written in the following forms

$$q^{[1]}(x,y,t) = q(x,y,t) - \frac{2i(\lambda_1 + \lambda_1^*)\psi_1(x,y,t)\psi_2^*(-x,-y,t)}{\Delta}, \tag{18}$$

$$v^{[1]}(x,y,t) = v(x,y,t) + 4i\left(\frac{\lambda_1 \psi_1(x,y,t)\psi_1^*(-x,-y,t) - \lambda_1^* \psi_2(x,y,t)\psi_2^*(-x,-y,t)}{\Delta}\right)_y, \tag{19}$$

with $\Delta = \psi_1 \psi_1^*(-x,-y,t) - \psi_2 \psi_2^*(-x,-y,t)$

So, in this section, we obtained the one-fold Darboux transformation of the nonlocal nonlinear Schrödinger-type equations (1)–(2). Next, we construct the analytical solutions for Eqs. (1)–(2).

Example 1. We take trivial "seed" solutions of Eqs. (1)–(2) as $q = 0, q^*(-x,-y,t) = 0, v = 0$. Then by substituting these seed solutions in Eqs. (7)–(8) we obtain system in the next form

$$\Psi_{1x} = -i\lambda\Psi_1, \tag{20}$$

$$\Psi_{2x} = i\lambda\Psi_2, \tag{21}$$

$$\Psi_{1t} = 2\lambda\Psi_{1y} - i\lambda\beta\Psi_1 + \frac{i\alpha}{2}\Psi_1, \tag{22}$$

$$\Psi_{2t} = 2\lambda\Psi_{2y} + i\lambda\beta\Psi_2 - \frac{i\alpha}{2}\Psi_2, \tag{23}$$

The system (20)-(23) can have the following solutions

$$\Psi_1 = e^{-i\lambda_1 x + i\mu_1 y + (-i\lambda_1\beta + \frac{i\alpha}{2} + 2i\lambda_1\mu_1)t}, \tag{24}$$

$$\Psi_2 = e^{i\lambda_2 x - i\mu_2 y + (i\lambda_2\beta - \frac{i\alpha}{2} - 2i\lambda_2\mu_2)t}, \tag{25}$$

with $\lambda_1 = a + bi$ $\mu_1 = c + id$, $\lambda_2 = -a + bi$ $\mu_2 = -c + id$ and d , a, b, c, are const. By substitution Eqs. (24)–(25) into Eqs. (18)–(19) the analytical solutions of the nonlocal nonlinear Schrödinger-type equations can be written as

$$q^{[1]}(x,y,t) = -\frac{4ai\,e^{\chi}}{e^{\theta_1} - e^{\theta_2}},$$

$$v^{[1]}(x,y,t) = 4\left(\frac{((a+bi)e^{\theta_1} + (a-bi)e^{\theta_2})}{e^{\theta_1} - e^{\theta_2}}\right)_y,$$

where

$$\theta_1 = (-4ad - 4bc + 2\beta b)t - 2iax + 2icy,$$

$$\theta_2 = -2iax + 2icy - 2tb\beta,$$

$$\chi = (\alpha + 4ac - 4bd)t - 2axi + 2bx + 2y(ic - d).$$

Conclusion

Thus, we studied the nonlocal nonlinear Schrödinger-type equations by presenting the Lax pair formulation. We found the one-fold Darboux transformation, and based on what we received the analytical solutions. The obtained result is significant and promising in the context of PT-symmetric systems because the real-valued conserved charges can be given physical meaning.

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ШРЕДИНГЕР ТИПТІ ЛОКАЛЬДЫ ЕМЕС СЫЗЫҚТЫҚ ЕМЕС ТЕҢДЕУЛЕРДІҢ АНАЛИТИКАЛЫҚ ШЕШІМДЕРІ

Аңдатпа

Физикада әртүрлі құбылыстарды сипаттау үшін сызықтық емес теңдеулер қолданылады. Әдетте, осы теңдеулер сызықтық емес дербес туынды дифференциалдық теңдеулер болып табылады. Олар Лакс жұптары деп аталатын екі сызықтық дифференциалдық теңдеулердің үйлесімділік шарттары ретінде қабылдануы мүмкін. Лакс жұбының болуы сызықтық емес дербес туынды дифференциалдық теңдеудің интегралдылығын анықтайды. Осы дамумен байланысты тор динамикасы, сызықты емес оптика және сұйықтық механикасы сияқты сызықты емес құбылыстарда негізгі рөл атқаратын солитондар деп аталатын белгілі бір когерентті құрылымдардың жүзеге асырылуы болды. Белгілі теңдеулердің бірі сызықты емес оптикадағы және Бозе-

Эйнштейн конденсатындағы әртүрлі физикалық құбылыстармен байланысты сызықты емес Шредингер теңдеуі болады. Бұл теңдеуде Лакс жұбы бар, сондықтан ол интегралданатын деп аталады. Бұл жұмыс PT симметриясы бар локальды емес сызықтық емес Шредингер типті теңдеулерді зерттейді. Локальды емес сызықтық емес теңдеулер әртүрлі физикалық контекстерде пайда болады, мысалы, гидродинамика, конденсацияланған күй физикасы, оптика және тағы басқа. Локальды емес сызықтық емес Шредингер типті теңдеулер үшін Лакс жұбы ұсынылған. Аналитикалық шешімдерді алу үшін Дарбу түрлендіру әдісі қолданылады.

Тірек сөздер: Шредингер типті теңдеулер, Дарбу түрлендіру, локальды емес, Лакс жұбы, аналитикалық шешім, симметрия.

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АНАЛИТИЧЕСКИЕ РЕШЕНИЯ НЕЛОКАЛЬНЫХ НЕЛИНЕЙНЫХ УРАВНЕНИЙ ТИПА ШРЁДИНГЕРА

Аннотация

В физике нелинейные уравнения применяются для описания различных явлений. Обычно нелинейные уравнения являются в виде нелинейных дифференциальных уравнений в частных производных. Эти уравнения можно принять как условие совместности двух линейных дифференциальных уравнений, называемых представлением Лакса, существование которых определяет интегрируемость нелинейного дифференциального уравнения в частных производных. С этим развитием было связано осознание того, что определенные когерентные структуры, известные как солитоны, играют фундаментальную роль в нелинейных явлениях, таких как динамика решетки, нелинейная оптика и механика жидкости. Одним из известных нелинейных уравнений является нелинейное уравнение Шрёдингера, которое связано с различными физическими явлениями в нелинейной оптике и конденсатах Бозе-Эйнштейна. Это уравнение допускает пару Лакса, поэтому оно интегрируемо. В данной работе исследуются нелинейные нелокальные уравнения типа Шрёдингера с PT-симметрией. Нелинейные нелокальные уравнения возникают в таких областях физики, как гидродинамика, физика конденсированного состояния, оптика и т.д. Представлена пара Лакса для нелинейных нелокальных уравнений типа Шрёдингера. Для получения аналитических решений применен метод преобразования Дарбу.

Ключевые слова: уравнения типа Шрёдингера, преобразование Дарбу, нелокальность, пара Лакса, аналитическое решение, симметрия.

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