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FACTORIZATIONS AND UNIFIED HARDY INEQUALITIES ON HOMOGENEOUS LIE GROUPS

Abstract

In this note we obtain Hardy and critical Hardy inequalities with any homogeneous quasi-norm in unified way. Actually, we show a sharp remainder formula for these results. In particular, our identity implies corresponding Hardy and critical Hardy inequalities with any homogeneous quasi-norm for the radial derivative operator, thus yielding improved versions of corresponding classical counterparts. Moreover, we discuss extensions of these results in the setting of Folland and Stein's homogeneous Lie groups. Such a more general setting is convenient for the distillation of those results of harmonic analysis depending only on the group and dilation structures, which is one of our motivations working in the setting. Our approach based on the factorization method of differential operators introduced by Gesztesy and Littlejohn. As an application, we show Caffarelli-Kohn-Nirenberg type inequalities with more general weight. Because of the freedom in the choice of any homogeneous quasi-norm, our results give new insights already in both anisotropic $\mathbb{R}^n$ and isotropic $\mathbb{R}^n$.

Key words: factorization method, Hardy inequality, homogeneous Lie group.

Introduction

Recall that the Hardy inequality for all functions $f \in C_0^\infty(\mathbb{R}^n)$ takes the form

$$\int_{\mathbb{R}^n} |(\nabla f)(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2} dx, \; n \geq 3, \tag{1}$$

where ∇ represents the standard gradient in $\mathbb{R}^n$. Nowadays, there are many works concerning various versions of Hardy's inequalities, so we are not able to cover it in this short paper. Let us mention only classic monographs such as [1], [2], [3], [4] and the recent open access book [5] on this subject and for its applications.

This note aims to establish more general Hardy inequality unifying both sub-critical and critical cases on homogeneous Lie groups by employing the factorization method for differential expressions.

In this direction, recently Gesztesy and Littlejohn in [6] gave a simple proof of (1) by using the non-negativity of $T_\alpha^+ T_\alpha$ on $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and selecting special value for the parameter α, where

$$T_\alpha := \nabla + \alpha |x|^{-2} x,$$

and T_α^+ its formal adjoint

$$T_\alpha^+ = -\mathrm{div}(\cdot) + \alpha |x|^{-2} x \,\cdot, \; x \in \mathbb{R}^n \setminus \{0\},$$

for $\alpha \in \mathbb{R}, x \in \mathbb{R}^n \setminus \{0\}$ and $n \geq 3$. There, the authors also applied the factorization method to obtain Rellich inequalities with remainder terms. For this method, we also refer to [7], [8], [9] in the Euclidean setting as well as [10] on stratified Lie groups.

Here, in this note we plan to continue the research from [10] by applying the method to unified Hardy inequalities on homogeneous Lie groups with any homogeneous quasi-norm. Actually, in this noncommutative setting we show Hardy identities that implies improved subcritical and critical Hardy inequalities. For this, let us recall first necessary definitions and notations of homogeneous Lie groups following Folland and Stein [11] and a recent book by Fischer and Ruzhansky [12].

A homogeneous Lie group $\mathbb{G}$ is a connected simply connected Lie group if its Lie algebra $\mathfrak{g}$ is equipped with family of morphisms representing dilations, as follows:

$$D_\lambda = \mathrm{Exp}(A \ln \lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} (\ln(\lambda) A)^k.$$

Here A is a diagonalisable positive linear operator on the Lie algebra $\mathfrak{g}$, and every D_λ is a morphism of the Lie algebra $\mathfrak{g}$. The homogeneous dimension of $\mathbb{G}$ is defined by $Q := \mathrm{Tr}\, A$. In this context, the standard Lebesgue measure dx on $\mathbb{R}^n$ is the Haar measure on homogeneous Lie group $\mathbb{G}$ [12, Proposition 1.6.6]).

A homogeneous quasi-norm refers to a continuous non-negative function

$$\mathbb{G} \ni \mathrm{x} \mapsto |x| \in [0, \infty)$$

satisfying:

- $|x^{-1}| = |x|$ for any $x \in \mathbb{G}$,
- $|\lambda x| = \lambda |x|$ for any $x \in \mathbb{G}$ and $\lambda > 0$,
- $|x| = 0$ if and only if $x = 0$.

Next, the polar decomposition on homogeneous Lie groups defined as follows: there exists a positive Borel measure σ defined on the unit quasi-sphere

$$\mathfrak{S} := \{x \in \mathbb{G} : |x| = 1\}, \tag{2}$$

so that for all $f \in L^1(\mathbb{G})$ we have

$$\int_{\mathbb{G}} f(x)dx = \int_0^\infty \int_{\mathbb{G}} f(ry) r^{Q-1} d\sigma(y) dr, \tag{3}$$

where $|\cdot|$ is a homogeneous quasi-norm on $\mathbb{G}$. For more details on the polar decomposition, we can refer to [11]. Throughout this paper, we also use the notation

$$\mathcal{R}_{|x|} := \frac{d}{d|x|} \tag{4}$$

for a homogeneous quasi-norm $|x|$ on the homogeneous Lie group $\mathbb{G}$.

Material and methods

As mentioned in Introduction there is a huge literature on the Hardy inequalities, so let us refer only to classic monographs as [1], [2], [3] and [4] to mention few. As for the factorization method, we can refer to [6], [7], [8], [9] and [10].

The main aim of this paper is to derive unified Hardy inequalities with any homogeneous quasi-norm on homogeneous Lie groups via the method of factorization. Gesztesy introduced the factorization method to obtain the classical Hardy inequality in [8] then in [9] the radial and logarithmic refinements. More recently, in the noncommutative setting, the authors in [10] successfully applied the factorization method to obtain improved Hardy and Hardy-Rellich type inequalities on general stratified groups and weighted Hardy inequalities on general homogeneous Lie groups.

Results and Discussion

Let $\mathrm{B}_R := \{y \in G : |y| < R\}$ where $|\cdot|$ is a homogeneous quasi-norm on $\mathbb{G}$.

Theorem 1. Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimensions $Q \geqslant 2$ and let $|\cdot|$ be any homogeneous quasi-norm. Let B_R be a quasi-ball of $\mathbb{G}$ with a radius R. Then, for all complex-valued functions $f \in C_0^\infty(\mathrm{B}_R \backslash \{0\})$ we have

Remark 1. By dropping the last term in (5), we can get the following result:

$$|\mathcal{R}_{|x|} f(x)|^2 dx = \frac{(Q-2)^2}{4} \int_{\mathrm{B}_R} \frac{|f(x)|^2}{|x|^2} dx + \frac{1}{4} \int_{\mathrm{B}_R} \frac{|f(x)|^2}{|x|^2} \left(\ln \frac{R}{|x|} \right)^{-2} dx$$
$$+ \int_{\mathrm{B}_R} \left| \mathcal{R}_{|x|} + \frac{Q-2}{2} |x|^{-1} + \frac{1}{2} |x|^{-1} \left(\ln \frac{R}{|x|} \right)^{-1} \right|^2 dx. \tag{5}$$

$$\int_{\mathrm{B}_R} |\mathcal{R}_{|x|} f(x)|^2 dx \geq \frac{(Q-2)^2}{4} \int_{\mathrm{B}_R} \frac{|f(x)|^2}{|x|^2} dx + \frac{1}{4} \int_{\mathrm{B}_R} \frac{|f(x)|^2}{|x|^2} \left(\ln \frac{R}{|x|} \right)^{-2} dx. \tag{6}$$

Here, note that the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, hence $Q = n$, of (6) with the Euclidean norm $|x| = |x|_E$ was obtained in [9]. So, the identity (5) gives sharp remainder formula for their result. Moreover, dropping the last term in (6) implies improved version of (1) due to the estimate $|\nabla f| \geq |\mathcal{R}_{|x|} f|$. Similarly, but dropping the first term instead of the second term on the right-hand side of (6) gives improved critical case $n = 2$ of (1).

Proof of Theorem 1. Let us first introduce one-parameter differential expression

$$T := \mathcal{R}_{|x|} + \frac{Q-2}{2} |x|^{-1} + \frac{1}{2} |x|^{-1} \left(\ln \frac{R}{|x|} \right)^{-1}. \tag{7}$$

Now, we find a formal adjoint operator of T on $C_0^\infty(\mathrm{B}_R \backslash \{0\})$ as follows:

$$\int_{\mathrm{B}_R} \mathcal{R}_{|x|} f(x) \overline{g(x)} dx + \frac{Q-2}{2} \int_{\mathrm{B}_R} |x|^{-1} f(x) \overline{g(x)} dx + \frac{1}{2} \int_{\mathrm{B}_R} |x|^{-1} \left(\ln \frac{R}{|x|} \right)^{-1} f(x) \overline{g(x)} dx$$
$$= \int_0^R \int_{\mathfrak{S}} \frac{d}{dr} (f(ry)) \overline{g(ry)} r^{Q-1} d\sigma(y) dr + \frac{Q-2}{2} \int_{\mathrm{B}_R} |x|^{-1} f(x) \overline{g(x)} dx$$
$$+ \frac{1}{2} \int_{\mathrm{B}_R} |x|^{-1} \left(\ln \frac{R}{|x|} \right)^{-1} f(x) \overline{g(x)} dx = - \int_0^R \int_{\mathfrak{S}} f(ry) \frac{d}{dr} (g(ry)) r^{Q-1} d\sigma(y) dr$$

$$-(Q-1)\int_0^R\int_{\mathfrak{S}} f(ry)\overline{g(ry)}r^{Q-2}d\sigma(y)dr$$

$$+\frac{Q-2}{2}\int_{B_R}|x|^{-1}f(x)\overline{g(x)}dx+\frac{1}{2}\int_{B_R}|x|^{-1}\left(\ln\frac{R}{|x|}\right)^{-1}f(x)\overline{g(x)}dx$$

$$=\int_{B_R}f(x)(-\overline{\mathcal{R}_{|x|}g(x)})dx-\frac{Q}{2}\int_{B_R}|x|^{-1}f(x)\overline{g(x)}dx$$

$$+\frac{1}{2}\int_{B_R}|x|^{-1}\left(\ln\frac{R}{|x|}\right)^{-1}f(x)\overline{g(x)}dx. \tag{8}$$

Thus, the formal adjoint operator of T has the following form

$$T^+=-\mathcal{R}_{|x|}-\frac{Q}{2}|x|^{-1}+\frac{1}{2}|x|^{-1}\left(\ln\frac{R}{|x|}\right)^{-1}. \tag{9}$$

Now, let us compute T^+T as follows.

$$(T^+Tf)(x)=-\mathcal{R}_{|x|}(\mathcal{R}_{|x|}f(x))-\mathcal{R}_{|x|}\left(\frac{Q-2}{2}|x|^{-1}f(x)\right)-\mathcal{R}_{|x|}\left(\frac{1}{2}|x|^{-1}\left(\ln\frac{R}{|x|}\right)^{-1}f(x)\right)$$

$$-\frac{Q}{2}|x|^{-1}\mathcal{R}_{|x|}f(x)-\frac{Q}{2}\left(\frac{Q-2}{2}\right)|x|^{-2}f(x)-\frac{Q}{4}|x|^{-2}\left(\ln\frac{R}{|x|}\right)^{-1}f(x)$$

$$+\frac{1}{2}|x|^{-1}\left(\ln\frac{R}{|x|}\right)^{-1}\mathcal{R}_{|x|}f(x)+\frac{Q-2}{4}|x|^{-2}\left(\ln\frac{R}{|x|}\right)^{-1}f(x)+\frac{1}{4}|x|^{-2}\left(\ln\frac{R}{|x|}\right)^{-2}f(x).$$

By the nonnegativity of T^+T, introducing the polar coordinate $(r,y)=\left(|x|,\frac{x}{|x|}\right)\in(0,\infty)\times\mathfrak{S}$ on $\mathbb{G}$, where $\mathfrak{S}$ is the quasi-sphere as in (2), and using (3), one calculates

$$0\leqslant\int_{B_R}|(Tf)(x)|^2dx=\int_{B_R}f(x)\overline{(T^+Tf)(x)}dx=$$
$$\operatorname{Re}\int_{B_R}f(x)\overline{(T^+Tf)(x)}dx=I_1+I_2+I_3+I_4+I_5, \tag{10}$$

where

$$I_1=-\operatorname{Re}\int_0^R\int_{\mathfrak{S}}f(ry)\overline{\frac{d}{dr}\left(\frac{d}{dr}(f(ry))\right)}r^{Q-1}d\sigma(y)dr,$$

$$I_2=-\frac{Q-2}{2}\operatorname{Re}\int_0^R\int_{\mathfrak{S}}f(ry)\overline{\frac{d}{dr}(r^{-1}f(ry))}r^{Q-1}d\sigma(y)dr,$$

$$I_3=-\frac{1}{2}\operatorname{Re}\int_0^R\int_{\mathfrak{S}}f(ry)\overline{\frac{d}{dr}\left(r^{-1}\left(\ln\frac{R}{r}\right)^{-1}f(ry)\right)}r^{Q-1}d\sigma(y)dr,$$

$$I_4 = -\frac{Q}{2}\mathrm{Re}\int_0^R \int_{\mathfrak{S}} f(ry)r^{-1}\overline{\frac{d}{dr}f(ry)}r^{Q-1}d\sigma(y)dr + \frac{1}{2}Re\int_0^R \int_{\mathfrak{S}} f(ry)r^{-1}\left(\ln\frac{R}{r}\right)^{-1}\overline{\frac{d}{dr}f(ry)}r^{Q-1}d\sigma(y)dr,$$

and,

$$I_5 = -\frac{Q}{2}\left(\frac{Q-2}{2}\right)\int_{B_R}\frac{|f(x)|^2}{|x|^2}dx - \frac{Q}{4}\int_{B_R}\frac{|f(x)|^2}{|x|^2}\left(\ln\frac{R}{|x|}\right)^{-1}dx$$

$$+\frac{Q-2}{4}\int_{B_R}\frac{|f(x)|^2}{|x|^2}\left(\ln\frac{R}{|x|}\right)^{-1}dx + \frac{1}{4}\int_{B_R}\frac{|f(x)|^2}{|x|^2}\left(\ln\frac{R}{|x|}\right)^{-2}dx. \qquad (11)$$

Now we simplify I_1, I_2, I_3 and I_4. By a direct calculation we obtain

$$I_1 = -\mathrm{Re}\int_0^R \int_{\mathfrak{S}} f(ry)\overline{\frac{d}{dr}\left(\frac{d}{dr}(f(ry))\right)}r^{Q-1}d\sigma(y)dr$$

$$= (Q-1)\,\mathrm{Re}\int_0^R \int_{\mathfrak{S}} f(ry)\overline{\frac{d}{dr}(f(ry))}r^{Q-2}d\sigma(y)dr$$

$$+\mathrm{Re}\int_0^R \int_{\mathfrak{S}} \frac{d}{dr}(f(ry))\overline{\frac{d}{dr}(f(ry))}r^{Q-1}d\sigma(y)dr =$$

$$= \frac{Q-1}{2}\int_0^R \int_{\mathfrak{S}} \frac{d}{dr}|f(ry)|^2 r^{Q-2}d\sigma(y)dr$$

$$+\int_0^R \int_{\mathfrak{S}} \left|\frac{d}{dr}(f(ry))\right|^2 r^{Q-1}d\sigma(y)dr$$

$$= -\frac{(Q-1)(Q-2)}{2}\int_0^R \int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}d\sigma(y)dr$$

$$+\int_0^R \int_{\mathfrak{S}} \left|\frac{d}{dr}(f(ry))\right|^2 r^{Q-1}d\sigma(y)dr$$

$$= \int_{B_R}|\mathcal{R}_{|x|}f(x)|^2dx - \frac{(Q-1)(Q-2)}{2}\int_{B_R}\frac{|f(x)|^2}{|x|^2}dx. \qquad (12)$$

Now, we calculate I_2 :

$$I_2 = -\frac{Q-2}{2}\mathrm{Re}\int_0^{\mathrm{R}}\int_{\mathfrak{S}} f(ry)\overline{\frac{d}{dr}\left(r^{-1}f(ry)\right)}r^{Q-1}d\sigma(y)dr$$

$$= \frac{Q-2}{2}\int_0^{\mathrm{R}}\int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}d\sigma(y)dr$$

$$-\frac{Q-2}{2}\mathrm{Re}\int_0^{\mathrm{R}}\int_{\mathfrak{S}} f(ry)\overline{\frac{d}{dr}f(ry)}r^{Q-2}d\sigma(y)dr$$

$$= \frac{Q-2}{2}\int_0^{\mathrm{R}}\int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}d\sigma(y)dr$$

$$-\frac{Q-2}{4}\int_0^{\mathrm{R}}\int_{\mathfrak{S}} \frac{d}{dr}|f(ry)|^2 r^{Q-2}d\sigma(y)dr$$

$$= \left(\frac{Q-2}{2}+\frac{(Q-2)^2}{4}\right)\int_0^{\mathrm{R}}\int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}d\sigma(y)dr$$

$$= \left(\frac{Q-2}{2}+\frac{(Q-2)^2}{4}\right)\int_{\mathrm{B}_R}\frac{|f(x)|^2}{|x|^2}dx. \tag{13}$$

For I_3, one has

$$\mathrm{I}_3 = -\frac{1}{2}\mathrm{Re}\int_0^{\mathrm{R}}\int_{\mathfrak{S}} f(ry)\overline{\frac{d}{dr}\left(r^{-1}\left(\ln\frac{R}{r}\right)^{-1}f(ry)\right)}r^{Q-1}d\sigma(y)dr$$

$$= \frac{1}{2}\int_0^{\mathrm{R}}\int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}\left(\ln\frac{R}{r}\right)^{-1}d\sigma(y)dr$$

$$-\frac{1}{2}\int_0^{\mathrm{R}}\int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}\left(\ln\frac{R}{r}\right)^{-2}d\sigma(y)dr$$

$$-\frac{1}{4}\int_0^{\mathrm{R}}\int_{\mathfrak{S}} \frac{d}{dr}|f(ry)|^2 r^{Q-2}\left(\ln\frac{R}{r}\right)^{-1}d\sigma(y)dr$$

$$= \frac{1}{2}\int_0^{\mathrm{R}}\int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}\left(\ln\frac{R}{r}\right)^{-1}d\sigma(y)dr$$

$$-\frac{1}{2}\int_0^{\mathrm{R}}\int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}\left(\ln\frac{R}{r}\right)^{-2}d\sigma(y)dr$$

$$+\frac{Q-2}{4}\int_0^{\mathrm{R}}\int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}\left(\ln\frac{R}{r}\right)^{-1}d\sigma(y)dr$$

$$+\frac{1}{4}\int_0^R \int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}\left(\ln\frac{R}{r}\right)^{-2} d\sigma(y)dr$$

$$=\frac{Q}{4}\int_0^R \int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}\left(\ln\frac{R}{r}\right)^{-1} d\sigma(y)dr$$

$$-\frac{1}{4}\int_0^R \int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}\left(\ln\frac{R}{r}\right)^{-2} d\sigma(y)dr$$

$$=\frac{Q}{4}\int_{B_R} \frac{|f(x)|^2}{|x|^2}\left(\ln\frac{R}{|x|}\right)^{-1} dx-\frac{1}{4}\int_{B_R} \frac{|f(x)|^2}{|x|^2}\left(\ln\frac{R}{|x|}\right)^{-2} dx. \qquad (14)$$

Finally for I_4 we obtain:

$$I_4=-\frac{Q}{2}\mathrm{Re}\int_0^R \int_{\mathfrak{S}} f(ry)r^{-1}\overline{\frac{d}{dr}f(ry)}r^{Q-1}d\sigma(y)dr$$

$$+\frac{1}{2}\mathrm{Re}\int_0^R \int_{\mathfrak{S}} f(ry)r^{-1}\left(\ln\frac{R}{r}\right)^{-1}\overline{\frac{d}{dr}f(ry)}r^{Q-1}d\sigma(y)dr$$

$$=-\frac{Q}{4}\int_0^R \int_{\mathfrak{S}} \frac{d}{dr}|f(ry)|^2 r^{Q-2}d\sigma(y)dr$$

$$+\frac{1}{4}\int_0^R \int_{\mathfrak{S}} \frac{d}{dr}|f(ry)|^2 r^{Q-2}\left(\ln\frac{R}{r}\right)^{-1} d\sigma(y)dr$$

$$=\frac{Q(Q-2)}{4}\int_0^R \int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}d\sigma(y)dr$$

$$-\frac{Q-2}{4}\int_0^R \int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}\left(\ln\frac{R}{r}\right)^{-1} d\sigma(y)dr$$

$$-\frac{1}{4}\int_0^R \int_{\mathfrak{S}} |f(ry)|^2 r^{Q-3}\left(\ln\frac{R}{r}\right)^{-2} d\sigma(y)dr$$

$$=\frac{Q(Q-2)}{4}\int_{B_R} \frac{|f(x)|^2}{|x|^2}dx-\frac{(Q-2)}{4}\int_{B_R} \frac{|f(x)|^2}{|x|^2}\left(\ln\frac{R}{|x|}\right)^{-1} dx$$

$$-\frac{1}{4}\int_{B_R} \frac{|f(x)|^2}{|x|^2}\left(\ln\frac{R}{|x|}\right)^{-2} dx. \qquad (15)$$

Putting (11)-(15) in (10) we obtain that

$$\int_{B_R} |\mathcal{R}_{|x|} f(x)|^2 dx - \frac{(Q-2)^2}{4} \int_{B_R} \frac{|f(x)|^2}{|x|^2} dx - \frac{1}{4} \int_{B_R} \frac{|f(x)|^2}{|x|^2} \left(\ln \frac{R}{|x|}\right)^{-2} dx$$

$$= \int_{B_R} |(Tf)(x)|^2 dx, \tag{16}$$

that is,

$$\int_{B_R} |\mathcal{R}_{|x|} f(x)|^2 dx =$$

$$\frac{(Q-2)^2}{4} \int_{B_R} \frac{|f(x)|^2}{|x|^2} dx + \frac{1}{4} \int_{B_R} \frac{|f(x)|^2}{|x|^2} \left(\ln \frac{R}{|x|}\right)^{-2} dx + \int_{B_R} |(Tf)(x)|^2 dx. \tag{17}$$

Caffarelli-Kohn-Nirenberg type inequalities on homogeneous Lie groups

Theorem 2. Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q. Let $|\cdot|$ be be any homogeneous quasi-norm on $\mathbb{G}$. Let $1 < q < \infty, 0 < r < \infty$ with $2 + q \geq r, \delta \in [0,1] \cap \left[\frac{r-q}{r}, \frac{2}{r}\right]$ and $b, c \in \mathbb{R}$. Assume that $\frac{\delta r}{2} + \frac{(1-\delta)r}{q} = 1$ and $c = \delta + b(1-\delta)$. Then for all $f \in C_0^\infty(B_R \backslash \{0\})$ we have the following Caffarelli-Kohn-Nirenberg type inequalities:

$$\|\omega^c f\|_{L^r(\mathbb{G})} \leq \left\|\mathcal{R}_{|x|} f\right\|_{L^2(\mathbb{G})}^{\delta} \|\omega^b f\|_{L^q(\mathbb{G})}^{1-\delta}, \tag{18}$$

where $\omega = \dfrac{\left(\frac{(Q-2)^2}{4} + \frac{1}{4}\left(\ln \frac{R}{|x|}\right)^{-2}\right)^{\frac{1}{2}}}{|x|}$.

Proof of Theorem 2. Case $\delta \in [0,1] \cap \left[\frac{r-q}{r}, \frac{2}{r}\right]$. Recalling $c = \delta + b(1-\delta)$, a direct calculation gives

$$\|\omega^c f\|_{L^r(\mathbb{G})} = \left(\int_{\mathbb{G}} |\omega|^{cr} |f(x)|^r dx\right)^{\frac{1}{r}} = \left(\int_{\mathbb{G}} \frac{|f(x)|^{\delta r}}{|\omega|^{-\delta r}} \cdot \frac{|f(x)|^{(1-\delta)r}}{|\omega|^{-br(1-\delta)}} dx\right)^{\frac{1}{r}}. \tag{19}$$

We want to use Holder's inequality for $\frac{\delta r}{2} + \frac{(1-\delta)r}{q} = 1$ and $2 + q \geq r$.

Then using Hölder's inequality in (19) we get following:

$$\|\omega^c f\|_{L^r(\mathbb{G})} \leq \left(\int_{\mathbb{G}} \frac{|f(x)|^2}{|\omega|^{-2}} dx\right)^{\frac{\delta}{2}} \left(\int_{\mathbb{G}} \frac{|f(x)|^q}{|\omega|^{-bq}} dx\right)^{\frac{1-\delta}{q}} \tag{20}$$

$$= \|\omega f\|_{L^2(\mathbb{G})}^{\delta} \left\|\frac{f}{\omega^{-b}}\right\|_{L^q(\mathbb{G})}^{1-\delta}.$$

From Theorem 1 we have

$$\left\|\mathcal{R}_{|x|}f\right\|_{L^2(\mathrm{B}_R)} \geq \left\|\frac{f(x)}{|x|}\left(\frac{(Q-2)^2}{4} + \frac{1}{4}\left(\ln\frac{R}{|x|}\right)^{-2}\right)^{\frac{1}{2}}\right\|_{L^2(\mathrm{B}_R)}. \quad (21)$$

Recalling $\omega(x) = \frac{\left(\frac{(Q-2)^2}{4}+\frac{1}{4}\left(\ln\frac{R}{|x|}\right)^{-2}\right)^{\frac{1}{2}}}{|x|}$, we can rewrite it as

$$\left\|\mathcal{R}_{|x|}f\right\|_{L^2(\mathrm{B}_R)} \geq \|\omega f\|_{L^2(\mathrm{B}_R)}. \quad (22)$$

Using (22) in (20) gives the desired inequality (18).

Conclusion

Finally, in this note we have established a unified result that contains both the critical and subcritical Hardy inequalities. Moreover, as a byproduct Caffarelli-Kohn-Nirenberg type inequality is obtained.

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ФАКТОРИЗАЦИЯ ЖӘНЕ БІРТЕКТІ ЛИ ТОПТАРЫНДАҒЫ БІРТҰТАС ХАРДИ ТЕҢСІЗДІГІ

Аңдатпа

Бұл мақалада біртекті квазинормалармен біртұтас Харди теңсіздіктері мен критикалық Харди теңсіздіктері зерттеледі. Жұмыста осы нәтижелер үшін нақты қалдық формулалары беріледі. Алынған теңдеулер классикалық аналогтардың жетілдірілген нұсқаларын қамтиды және радиалды туынды операторы үшін кез келген біртекті квазинормамен сәйкес келетін Харди теңсіздіктері мен критикалық Харди теңсіздіктерін қамтиды. Сонымен қатар, Фолланд пен Стейннің біртекті Ли топтарының құрылымындағы нәтижелерді кеңейтуді талқылаймыз. Жалпылама тұжырым тек топтық және кеңейтілген құрылымдарға тәуелді гармоникалық талдау нәтижелерін жалпылау үшін ыңғайлы және бұл бағытта жұмыс істеудегі басты мотивацияларымыздың бірі. Жұмыстағы негізгі әдіс Гестези мен Литтлджон ұсынған дифференциалды операторды факторизациялау әдісіне негізделген. Қолданысы ретінде, жалпы салмақтары бар Каффарелли-Кон-Ниренберг типті теңсіздіктер көрсетілген. Кез келген біртекті квазинорманы еркін таңдаудың арқасында біздің нәтижелеріміз маңызды артықшылықтарға ие, өйткені олар $\mathbb{R}^n$ анизотропты кеңістігіне де, $\mathbb{R}^n$ изотропты кеңістігіне де қолданылады.

Тірек сөздер: факторизация әдісі, Харди теңсіздігі, біртекті Ли тобы.

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ФАКТОРИЗАЦИЯ И ЕДИНООБРАЗНЫЕ НЕРАВЕНСТВА ХАРДИ НА ОДНОРОДНЫХ ГРУППАХ ЛИ

Аннотация

В этой работе мы получаем неравенства Харди и критические неравенства Харди с любой однородной квазинормой единым способом. На самом деле мы показываем формулу точного остатка для этих результатов. В частности, наше тождество подразумевает соответствующие неравенства Харди и критические неравенства Харди с любой однородной квазинормой для оператора радиальной производной, что дает улучшенные версии соответствующих классических аналогов. Более того, мы обсуждаем расширение этих результатов в контексте однородных групп Ли Фолланда и Штейна. Такая более общая постановка удобна для обобщения результатов гармонического анализа, зависящих только от групповой и расширяющих структур, что является одной из наших мотиваций при работе в этой среде. Наш подход основан на методе факторизации дифференциальных операторов, предложенном Гестези и Литтлджоном. В качестве приложения мы показываем неравенства типа Каффарелли-Кона-Ниренберга с более общим весом. Благодаря свободе выбора с любой однородной квазинормы наши результаты уже дают значительные преимущества, так как они применимы как к анизотропному пространству $\mathbb{R}^n$, так и к изотропному пространству $\mathbb{R}^n$.

Ключевые слова: метод факторизации, неравенство Харди, однородная группа Ли.

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