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COMPLETE CLASSIFICATION OF QUADRATIC IRRATIONALS WITH PERIOD TWO

Abstract

This article presents a comprehensive investigation into the classification of quadratic irrationals with period two in their continued fraction representations. Building upon foundational results in Number Theory, particularly in the context of continued fractions and Pell's equation, the study reveals intricate relationships between quadratic irrationals and their periodic structures. The main object of study is $\sqrt{N}$ and properties of its continued fractions. While it is well-known that continued fractions of $\sqrt{N}$ is periodic with periodic part being palindrome, the distribution of the lengths of the periodic parts are far from being complete. Our main goal will be to focus on the period two case and provide a complete characterization. The research's proved theorems clarify the conditions under which the period length is exactly two and give an insight into the underlying algebraic features. Additionally, it delves deeper by offering numerical analysis and illustrations demonstrating the distribution of period lengths among quadratic irrationals. This research opens up new paths for future studies on quadratic irrationals and how they're shown as continued fractions.

Key words: Number Theory, continued fractions, quadratic irrationals, Pell’s equation, period lengths.

Introduction

The theory of continued fractions is a rich branch of Number Theory that considers the representation of real numbers x by infinite sequences of fractions:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

Continued fractions have a wide range of applications, including in numerical stability and special function representation [1], number-theoretic computations [2] and special functions in various fields [3]. They are also used in convergence criteria, three-term recurrence relations, hypergeometric functions, Pade approximants, and zero-free regions [4]. These applications highlight the versatility and importance of continued fractions in various mathematical and computational fields.

In this work, we consider quadratic irrationals and study the lengths of their periods when represented as continued fractions. We say that an irrational numbers is quadratic irrational if it is a root of a quadratic polynomial with integer coefficients. For simplicity we write $[a_0; a_1, a_2, a_3, \dots]$ for the continued fraction representation. One of the important results in the theory of continued fractions is due to Lagrange which states that for any quadratic irrational number, the continued fraction expansion eventually becomes periodic, for a proof see e.g. [5]. A special case of quadratic irrationals is the square roots of integers $\sqrt{N}$. An amazing result due to Galois states that the periodic parts of such fractions for quadratic irrationals manifest as palindromes [6]. To me more precise, for any square free positive integer N there exist positive number $a_0, a_1, \dots, a_n$ such that

$$\sqrt{N} = [a_0, \overline{a_1, a_2, \dots, a_2, a_1, 2a_0}],$$

where the overlined is the periodic part that repeats in the continued fraction representation. One interesting research direction is to investigate the distributional properties of the quadratic irrationals. For any square free N we let $D(\sqrt{N})$ denote the length of the periodic part of the continued fraction for the quadratic irrational $\sqrt{N}$. In this line of research, a recent work [7] shows that that for any square free d the sequence

$$D(\sqrt{d}), D(2\sqrt{d}), D(3\sqrt{d}), D(4\sqrt{d}), \dots$$

has infinitely many limit points in discrete topology. While this result is a moderate contribution to the research area, there are still many interesting questions to investigate. In particular, their result is not constructive in the sense that it is not clear which periods appear infinitely often. This is a delicate question and to address one needs to classify the numbers with certain periods.

In this work, we aim to classify all the square free positive integers N such that $D(\sqrt{N}) = 2$. By employing the fundamental relationship between the length of the periodic part and Pell's equation, this research aims to bridge the gap between historical insights and contemporary research. It seeks to affirm the presence of an infinite set of positive integers for which the period length in the continued fraction representation of quadratic irrationals is exactly two, thereby contributing to the broader discourse on Number Theory and the intricate beauty of mathematical structures.

We now state our main results. The core of this work is encapsulated in a series of theorems that shed a light on the intricate relationship between quadratic irrationals and their continued fraction representations. The following result provides a criterion for a square root of a number to have period 2.

Theorem 1. The length of the period for the continued fraction of the quadratic irrational $\sqrt{N}$ is $D(\sqrt{N}) = 2$ if and only if there are positive integers a, b satisfying

$$N = a^2 + \frac{2a}{b}$$

such that b is divisible by a, but $2a \neq b$.

To illustrate the theorem, for example, we may take $a = b = 2$, then all the conditions of the theorem are satisfied and for $N = a^2 + \frac{2a}{b} = 6$ we see that the continued fraction expansion satisfies $\sqrt{6} = [2; 2, 4, 2, 4, 2, 4, 2, 4, \dots]$ giving $D(\sqrt{6}) = 2$. As another example, we may take $a = 4, b = 2$ then $N = a^2 + \frac{2a}{b} = 20$ and $D(20) = 2$.

The next result deals with those members of the sequence $(n\sqrt{d})$ for which the period length is 2.

Theorem 2. Let d be a square free positive integer. Then, the length of the period for the continued fraction of the quadratic irrational $n\sqrt{d}$ is $D(n\sqrt{d}) = 2$ if n satisfies the Pell's equation

$$n^2 - dx^2 = 1$$

for some integer x. In particular, there exist infinitely many n such that $(n\sqrt{d}) = 2$.

As a corollary to Theorem 2 we deduce the following distribution result.

Corollary 3. Let d be a square free positive integer. Then, the natural number n such that $D(n\sqrt{d}) = 2$ satisfies

$$\text{limsup}_{N\to\infty} \frac{1}{\ln N} |\{n \mid 1 \le n \le N, D(n\sqrt{d}) = 2\}| > 0.$$

These findings not only deepen our understanding of mathematical properties of quadratic irrationals but also establish a connection with Pell's equation, highlighting a significant intersection between different areas of Number Theory.

Collectively, these theorems illuminate mathematical intricacies of continued fractions for quadratic irrationals, bridging historical insights with contemporary mathematical inquiries. They demonstrate the rich interplay between algebraic forms, periodicity, and Diophantine equations, paving the way for future research in the field. Through detailed proofs and illustrative examples, this research underscores the enduring allure and complexity of continued fractions, inviting a further exploration into the mysteries of mathematical structures.

In the next section we review background and related works. Section 3 is devoted proving the main results. The paper ends with conclusion and future directions.

Main Provisions

The study of continued fractions, integral to mathematics, relies on historical achievements for advancement. Originating over two millennia ago, their formal foundations were laid in the late 17th and early 18th centuries. The genesis of continued fractions is often linked to Euclid's Algorithm, initially devised for finding the greatest common denominator (GCD) of two numbers but also applicable algebraically to derive the simple continued fraction representation of a rational number [8].

Euclid's method, when applied to find the GCD of 27 and 129, reveals a continued fraction representation through algebraic manipulations. This method highlights the reciprocal relationship between the fractions in successive equations, leading to a continued fraction representation. This early connection to continued fractions underscores their historical significance in approximating quadratic irrationals and contributions by mathematicians like Fibonacci.

During the Renaissance, mathematicians including Rafael Bombelli and John Wallis contributed significantly to the formal concept of continued fractions. Bombelli, in his treatise "L'Algebra" (1572), discussed square roots' representation as infinite continued fractions, providing a foundational approach later elaborated in his second edition of "L'Algebra Opera" (1579) [9]. Wallis introduced a continued fraction notation in "Arithmetica Infinitorum" (1655), using it to approximate square roots and other irrational numbers, thus coining the term continued fraction [9, 10].

The 18th and 19th centuries witnessed substantial advancements in continued fractions by Euler, Lagrange, and Gauss. Euler's work, "De fractionibus continuis" (1737), proved that every real number has a unique simple continued fraction representation and explored continued fractions for rational and irrational numbers, including the representation of e as a continued fraction [10]. Huygens applied convergents in continued fractions for practical uses, like gear ratio approximations in his mechanical planetarium construction [10, 11].

Lagrange's "Théorie des fonctions analytiques" (1798) established the periodicity of continued fractions for quadratic irrationals, known as Lagrange's theorem, which significantly influenced the development of periodic continued fractions [12]. The 19th century saw applications of continued fractions in Number Theory and Diophantine approximation by Cauchy and Sylvester, enhancing the understanding of partial quotients and approximation properties.

In 1972, P. Chowla and S. Chowla explored periodic continued fractions, raising questions about finding infinite sets of positive integers d with a given length of the periodic part $D\sqrt{d} = l$, conjecturing an affirmative answer [13]. Christian Friesen (1988) provided a positive proof, establishing results on palindromic sequences that led to Friesen's corollary addressing the original question posed by Chowla and Chowla [14].

Franz Halter-Koch (1989) refined Friesen's theorem, adding conditions related to the prime factorizations of d, offering a deeper understanding of the integers satisfying the palindromic sequence condition [15]. Recent studies by Balková and Hrusková compiled findings on continued fractions for quadratic numbers, investigating equations linking N to values of $D(\sqrt{N})$, contributing to the broader understanding of continued fractions in Number Theory [14].

Rada and Starosta explored the Moebius transformation's effect on the period length of continued fraction expansions, setting bounds for $D\left(h(\sqrt{x})\right)$ based on $D(\sqrt{x})$,

enriching the mathematical exploration of continued fractions [16]. Building on the work of Gawron and Kobos, this dissertation extends the analysis of Moebius transformations applied to quadratic irrationals, particularly focusing on their limit points in the sequence

$D(n\sqrt{d})$, highlighting the infinite nature of these limit points and posing new research directions.

Materials and Methods

Proof of Theorem 1. Assume that $D(\sqrt{N}) = 2$. By utilizing the definition of continued fractions and leveraging Galois' result which states that the periodic part of continued fractions for quadratic irrationals is a palindrome, we can represent $\sqrt{N}$ as follows:

$$\sqrt{N} = [a; b, 2a, b, 2a, b, 2a, \dots].$$

We notice that to avoid period 1 case, we must have $b \neq 2a$. Now, let's add the integer a to both sides of the expression and introduce a new equation:

$$x := a + \sqrt{N} = [2a; b, 2a, b, 2a, b, 2a, \dots] = [\overline{2a, b}].$$

Since $x = [\overline{2a, b}] = [2a, b, \overline{2a, b}]$, it follows that

$$x = 2a + \cfrac{1}{b + \cfrac{1}{x}} = 2a + \frac{x}{bx + 1}.$$

From this equation we can derive the quadratic equation in the following form:

$$x^2 - 2ax - \frac{2a}{b} = 0.$$

By solving this quadratic equation, we obtain two roots and derive the formula for the integer N:

$$x_1 = \frac{2a + \sqrt{4a^2 - 8a/b}}{2} = a + \sqrt{a^2 + \frac{2a}{b}},$$

$$x_2 = \frac{2a - \sqrt{4a^2 - 8a/b}}{2} = a - \sqrt{a^2 + \frac{2a}{b}}.$$

Since $x = a + \sqrt{N} > a$, we see that $x = x_1$. Solving for N we arrive at

$$N = a^2 + \frac{2a}{b}.$$

Therefore, we deduce that $D(\sqrt{N}) = 2$ if and only if $b \neq 2a$ and b is a positive integer divisible by $2a$.

We now turn in proving Theorem 2.

Proof of Theorem 2. Let a square free d be given. For $N = n\sqrt{d}$ to have period 2 we know from Theorem 1 that there must exist a, b such that $b \neq 2a$ and b is a positive integer divisible by $2a$ and

$$n^2 d = a^2 + \frac{2a}{b}.$$

Let us take $a = dx$ and $b = 2x$ where x is yet to be determined. Then, the above equation takes the form $n^2 d = (dx)^2 + d$. Cancelling out d we arrive at

$$n^2 - dx^2 = 1.$$

This is a well-known Pell's equation and it has infinitely many solution pairs (n, x) [17]. Therefore, for any such n we have that $D(n\sqrt{d}) = 2$. This finishes the proof of Theorem 2.

Proof of Corollary 3. We notice that once the fundamental solution (n_1, x_1) to the Pell's equation

$$n^2 - dx^2 = 1$$

is found, the other solutions (n_k, x_k) can be found iteratively from

$$n_k + x_k\sqrt{d} = \left(n_1 + x_1\sqrt{d}\right)^k.$$

Since $x_1\sqrt{d} = n_1^2 - 1$ we get

$$n_k \leq n_k + x_k\sqrt{d} = \left(n_1 + x_1\sqrt{d}\right)^k < (2n_1^2)^k.$$

Thus, for $N = (2n_1^2)^k$, there are at least k integers n satisfying $D(n\sqrt{d}) = 2$. Hence,

$$\limsup_{N\to\infty} \frac{1}{\ln N} |\{n \mid 1 \leq n \leq N, D(n\sqrt{d}) = 2\}| \geq \frac{1}{2n_1^2}.$$

Results and Discussion

In this section we carry numerical analysis and visualize distribution of lengths of periods to certain generality.

In the first instance, we consider all square-free positive integers up to one million and consider how the lengths of periods of under the square root function is distributed.

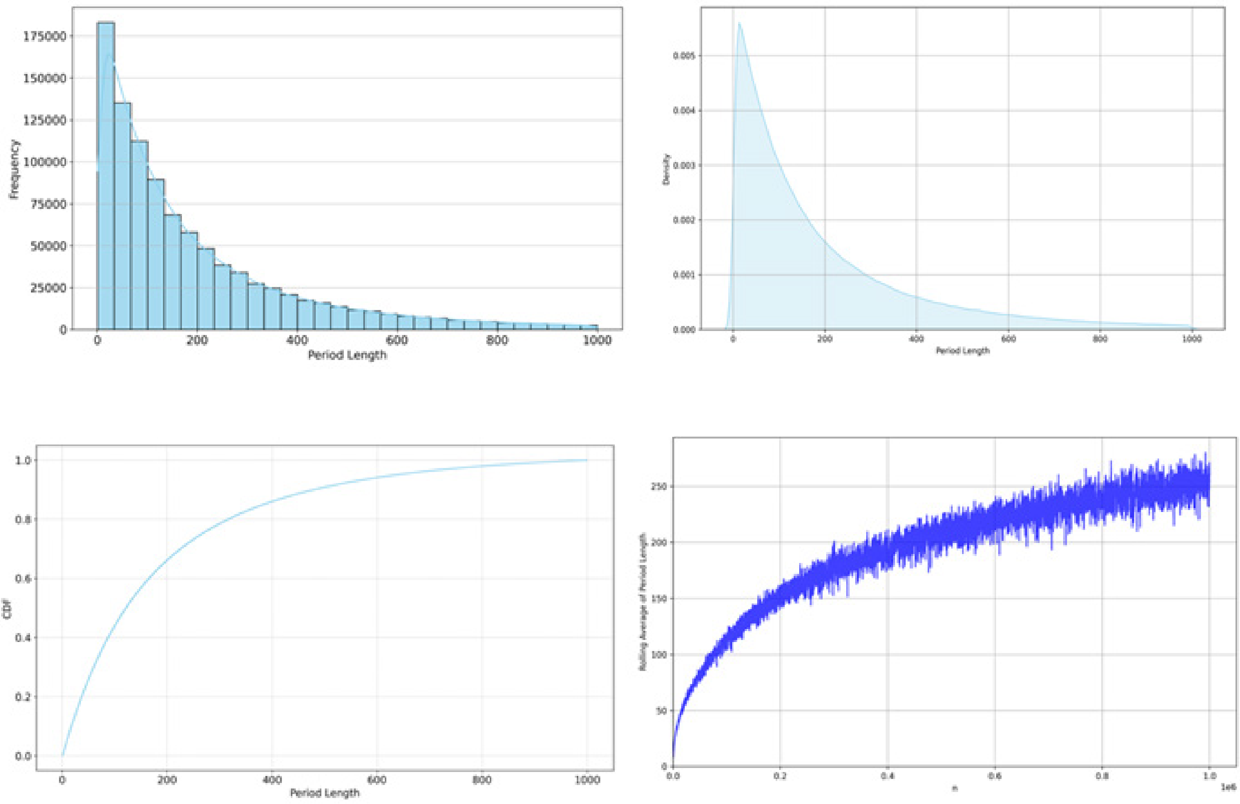


Figure 1 – Distribution of lengths of periods for square root
of numbers up to one million

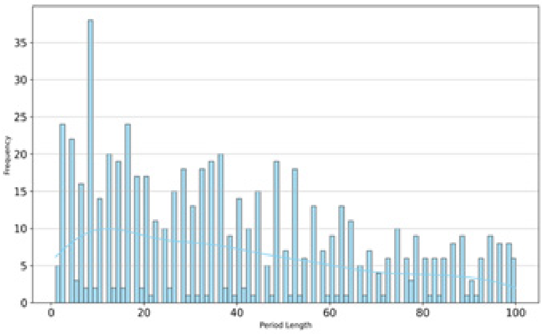
For most square-free $N \leq 1000000$ we have $D(N) \leq 1000$. For this reason, we have only considered those lengths that are at most 1000 to make the graphics readable.

In Figure 2, we consider the distribution of $D(n\sqrt{d})$ for various d and values for n up to 1000. The distribution of periods is generated when $d = 2,3,5,7,8,10$.

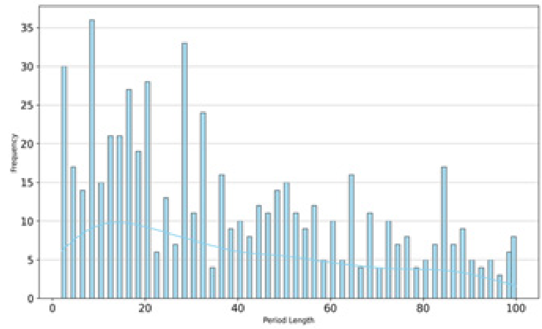
The implications of our findings are manifold, extending beyond the immediate scope of quadratic irrationals to touch upon broader aspects of number theory and mathematical research. The classification of quadratic irrationals with a period of two offers a glimpse into the symmetry and patterns inherent in mathematical structures, echoing the palindromic beauty found within the periodic parts of continued fractions for quadratic irrationals.

Looking ahead, several avenues for future research have been illuminated by this study. For instance, extending the classification to quadratic irrationals with periods greater than two presents a challenging yet potentially rewarding endeavor. Additionally, exploring the connections between the distribution of period lengths and other number-theoretic phenomena could yield further insights into the underlying principles of number theory.

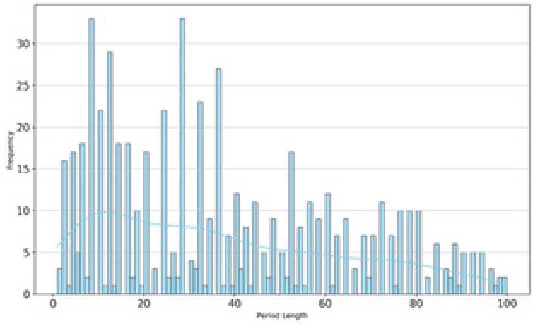
Another intriguing direction is the investigation of the practical applications of these findings. The theoretical insights gained from the study of quadratic irrationals and their continued fraction representations could have implications for numerical analysis, cryptography, and even quantum computing, where the properties of numbers play a pivotal role.



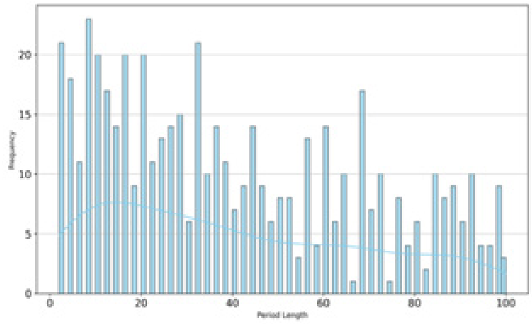
(A) $d = 2$



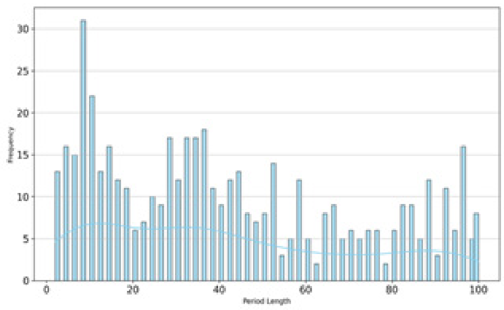
(B) $d = 3$



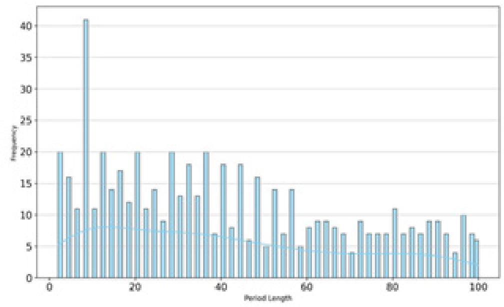
(C) $d = 5$



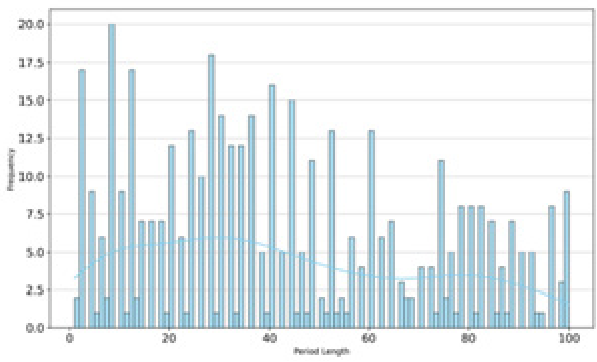
(D) $d = 6$



(E) $d = 7$



(F) $d = 8$



(G) $d = 10$

Figure 2 – The distribution of $D(n\sqrt{d})$ for $n = 1,2,\ldots,1000$ and $d = 2,3,5,6,7,8,10$.

Conclusion

This study delved into the realm of quadratic irrationals with period two, illuminating their structure through the lens of continued fractions and Pell's equation. Our work not only enriches the existing body of knowledge but also draws connections between classical and contemporary number theory, highlighting the intricate dance between algebraic properties and periodicity. The theorems we've established lay down explicit criteria for when a quadratic irrational will exhibit this unique periodic behavior, thereby expanding our understanding and appreciation of these mathematical entities.

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REFERENCES

1 Jones W.B., Thron W.J. Numerical stability in evaluating continued fractions. Mathematics of Computation, 1974, vol. 28, no. 127, pp. 795–810.

2 Williams H.C. Continued fractions and number-theoretic computations. The Rocky Mountain Journal of Mathematics, 1985, pp. 621–655.

3 Cuyt A.A.M. et al. Handbook of continued fractions for special functions. Springer Science & Business Media, 2008.

4 Lorentzen L., Waadeland H. Continued Fractions: Convergence Theory. Atlantis Press, 2008, vol. 1.

5 Einsiedler M. Ergodic theory. Springer, 2011.

6 Halter-Koch F. Quadratic irrationals: An introduction to classical number theory. – CRC press, 2013.

7 Gawron F., Kobos T. On length of the period of the continued fraction of nd. International Journal of Number Theory, 2023, pp. 1–11.

8 Van Tuyl A. L. An Introduction to the Theory and Applications of Continued Fractions, 1996.

9 Olds C. D. Continued Fractions Random House. New York, 1963.

10 Schlapp R. Morris Kline, Mathematical Thought from Ancient to Modern Times (Oxford University Press, 1973), xvii+ 1238 pp., £ 12. Proceedings of the Edinburgh Mathematical Society, 1973, vol. 18, no. 4, pp. 340–341.

11 Moore C. G. An Introduction to Continued Fractions, 1964.

12 De Lagrange J. L. Recherches d'arithmétique. Nouveaux Mémoires de l'Académie de Berlin, p. 1773.

13 Chowla P., Chowla S. Problems on periodic simple continued fractions. Proceedings of the National Academy of Sciences, 1972, vol. 69, no. 12, pp. 3745–3745.

14 Friesen C. On continued fractions of given period. Proceedings of the American Mathematical Society, 1988, vol. 103, no. 1, pp. 9–14.

15 Halter-Koch F. Continued fractions of given symmetric period. Fibonacci Quart.,1991, vol. 29, no. 4, pp. 298–303.

16 Řada H., Starosta Š. Bounds on the period of the continued fraction after a möbius transformation. Journal of Number Theory, 2020, vol. 212, pp. 122–172.

17 Acewicz M., Pąk K. Pell's equation. Formalized Mathematics, 2017, vol. 25, no. 3, pp. 197–204.

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ЕКІ ПЕРИОДТЫ КВАДРАТТЫҚ ИРРАЦИОНАЛ САНДАРДЫҢ ТОЛЫҚ ЖІКТЕЛУІ

Аңдатпа

Бұл мақалада квадраттық иррационал сандардың жіктелуін олардың үздіксіз бөлшек көріністерінде екінші кезеңмен жан-жақты зерттеу ұсынылған. Сандар теориясының негізгі нәтижелеріне сүйене отырып тізбекті фракциялар мен Пелл теңдеулерін зерттеу квадраттық иррационал сандар мен олардың периодтық құрылымдары арасындағы күрделі қатынастарды айқындайды. Зерттеудің негізгі объектісі – $\sqrt{N}$ және оның тізбекті бөлшектерінің қасиеттері. $\sqrt{N}$ үздіксіз бөлшектері периодты және олардың периодтық бөлігі палиндром екені белгілі болғанымен, периодтық бөліктердің ұзындығының таралуы толық зерттелмеген. Біздің басты мақсатымыз екінші кезеңге назар аудара отырып, оларға толық сипаттама беру. Зерттеуде дәлелденген теоремалар периодтың ұзындығы екіге тең болатын шарттарды нақтылайды және негізгі алгебралық белгілер туралы түсінік береді. Сонымен қатар, жұмыс квадраттық иррационал сандар арасында периодтық ұзындықтардың таралуын көрсететін сандық талдаулар мен иллюстрацияларды ұсына отырып, осы саладағы зерттеулерді тереңдете түседі. Бұл зерттеу болашақта квадраттық иррационал сандарды және олардың үздіксіз бөлшектер ретінде көрінуін зерттеуге жаңа жолдар ашады.

Тірек сөздер: Сандар теориясы, жалғасымды бөлшектер, квадрат иррационалдар, Пелл теңдеуі, период ұзындықтары

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ПОЛНАЯ КЛАССИФИКАЦИЯ КВАДРАТИЧНЫХ ИРРАЦИОНАЛЬНЫХ ЧИСЕЛ С ПЕРИОДОМ ДВА

Аннотация

В этой статье представлено всестороннее исследование классификации квадратичных иррациональных чисел со вторым периодом в их представлениях непрерывной дроби. Основываясь на фундаментальных результатах теории чисел, особенно в контексте цепных дробей и уравнения Пелла, исследование раскрывает сложные взаимосвязи между квадратичными иррациональными числами и их периодическими структурами. Основным объектом исследования является $\sqrt{N}$ и свойства его цепных дробей. Хотя хорошо известно, что непрерывные дроби $\sqrt{N}$ являются периодическими, а периодическая часть является палиндромом, распределение длин периодических частей далеко не полное. Нашей главной целью будет сосредоточиться на втором периоде и предоставить полную характеристику. Доказанные теоремы исследования разъясняют условия, при которых длина периода равна ровно двум, и дают представление о лежащих в основе алгебраических особенностях. Кроме того, он углубляется, предлагая численный анализ и иллюстрации, демонстрирующие распределение длин периодов среди квадратичных иррациональных чисел. Это исследование открывает новые пути для будущих исследований квадратичных иррациональных чисел и того, как они отображаются в виде непрерывных дробей.

Ключевые слова: теория чисел, цепные дроби, квадратичные иррациональные числа, уравнение Пелла, длины периодов.

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