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ENHANCED ALGORITHM FOR COMPUTING CAYLEY'S FIRST HYPERDETERMINANT

Abstract

Combinatorial hyperdeterminant DET – is the homogeneous polynomial in the entries of a hypermatrix of even number of indices, which is also a unique SL-invariant of minimal degree. It was first studied by Cayley in the middle of 19-th century. Given its fundamental nature, the computation of this polynomial is an important task. For fixed d and a cubical hypermatrix X of length n Barvinok introduced an algorithm of computing hyperdeterminant in $\mathrm{O}(2^{\mathrm{nd}}\mathrm{n}^{\mathrm{d}-1})$. Since the problem of deciding whether for the given hypermatrix X the hyperdeterminant $\mathrm{DET}(\mathrm{X})$ is equal to zero is NP-hard, it is essential to develop efficient algorithm for computing hyperdeterminant, as the size of problem grows exponentially. We provide enhanced algorithm of computing hyperdeterminant that requires $\mathrm{O}(2^{\mathrm{n}(\mathrm{d}-1)}\mathrm{n}^{\mathrm{d}-1})$ arithmetic operations.

Key words: Cayley's first hyperdeterminant, SL-invariant, Laplace expansion.

Introduction

The concept of hyperdeterminants has its roots in the study of multidimensional matrices or hypermatrices, which are generalizations of the conventional two-dimensional matrices to higher dimensions. The notion was coined by Arthur Cayley in the mid of 19-th century [1, 2], where he introduced first generalization of ordinary determinant, which is now referred to Cayley's first or combinatorial hyperdeterminant. Formally, let $T_d(n)$ denotes the space of tensors

$$T_d(n) := \mathbb{C}^n \otimes \mathbb{C}^n \otimes \ldots \otimes \mathbb{C}^n$$

i.e. d – dimensional hypermatrices. See [10] for extensive review. Throughout the paper d regarded as even positive integer number. Define a function of a hypermatrix $\{X(i_1, \ldots i_d)\}_{i_k \in [1,n]} = X \in T_d(n)$ as follows:

$$DET(X) := \frac{1}{n!} \sum_{\sigma_1 \in S_n, \sigma_2 \in S_n, \ldots, \sigma_d \in S_n} \operatorname{sgn}(\sigma_1 \ldots \sigma_d) \prod_{i=1}^{n} X\big(\sigma_1(i), \ldots, \sigma_d(i)\big), \tag{1}$$

where S_n denotes the set of permutations of length n.

While determinants of two-dimensional matrices have been extensively studied and utilized in various mathematical and physical applications, the exploration of hyperdeterminants in the realm of higher-dimensional arrays is relatively nascent yet profoundly significant. For instance, the computation of a determinant of an ordinary $n \times n$ matrix requires $O(n^3)$ arithmetic operations, i.e. polynomial time. Some properties of ordinary determinant can be generalized to the context of hyperdeterminant, such as Binet-Cauchy formula or Laplace expansion. On the other hand, other concepts cannot be easily described: the rank of hypermatrix or the zero locus (set of hypermatrices that nulls) of combinatorial hyperdeterminant.

In [3] Barvinok showed the algorithm of computing hyperdeterminant in $O(2^{nd}n^{d})$ arithmetic operations, that we refer in this paper as Algorithm 1. In this paper, we provide an enhancement of that algorithm, that we refer to as Algorithm 2.

Theorem 1. (Main theorem) Algorithm 2 computes the hyperdeterminant of a given d-dimenisonal hypermatrix A of length n in $O\left(2^{n(d-1)}n^{d-1}\right)$ arithmetic operations.

Presented algorithm works faster in 2^{n} times which is an essential optimization, since the size of a problem grows exponentially with growth of n or d. Since there is no hope for computing hyperdeterminant faster than the exponential time (since most tensor problems are NP-hard [4]), any enhancement of known algorithms can make a difference.

The study of hyperdeterminants was initiated by Arthur Cayley in the middle of 19-th centure [1, 2]. He first discovered combinatorial hyperdeterminant $DET(X)$, which nowadays referred to as Cayley's first hyperdeterminant for even d, along with Cayley's second hypdeterminant of a $2\times2\times2$ hypermatrices. In later papers, he refers to any $SL(n)^{d}$ invariant polynomial as hyperdeterminant.

In 1990s Gelfand-Kapranov-Zelevinsky introduced generalization of Cayley's second hyperdeterminant [5], which possess geometric properties of ordinary determinant and Cayley's second hyperdeterminant, which is referred to as geometric hyperdeterminant. Gemotric hyperdeterminant finds connections in various areas. For instance, it sheds light on the structure of tensors spaces [18]. Nevertheless, from computational point of view, geometric hyperdeterminant is infeasible, since the degree grows exponentionally as n or d grows. Moreover, no general formula is known.

The general problem of deciding if combinatorial hyperdeterminant vanishes is NP-hard [4], yet for some classes of (relatively sparse) tensors there are polynomial time algorithms [9]. See also [19] for exposition on tensors in computations.

In general, study of invariants of tensors is an important problem. For instance, discoveries of invariant theory may affect the asymptotic of matrix multiplication, see [14], see [16] for review of the subject. Recent advancements in the complexity of isomorphism problems have provided new insights into the computational challenges associated with tensor computations. In [11] authors introduce the concept of tensor isomorphism-completeness, establishing that many isomorphism problems for tensors are as hard as the general isomorphism problem for tensors.

In turn, combinatorial hyperdeterminant is of minimal degree n (i.e. multi-linear function) and has a simple formula. In 20-th century, some research were established to lift proprties of ordinary determinant to hyperdeterminant. In particular, due to its linear nature, several properties were obtained [6, 7], such as Laplace Expansion, Minor summation formula and Binet-Cauchy formula.

This hyperdeterminant has wide range of applications. Luque and Thibon [7] explored hyperdeterminants within the context of Selberg's and Aomoto's integrals, using purely algebraic methods to evaluate hyperdeterminants of Hankel type. This work highlighted the broader applicability of hyperdeterminants in multidimensional integrals and reinforced their role in mathematical physics. In [15], within the context of SYK model with one time point, half-wormhole contribution in factorization of decoupled systems is expressed through a hyperpfaffian (which is closesely related to hyperdeterminant) of the tensor of SYK couplings. It is also related to characteristic polynomials of hypergraphs [20].

In combinatorics, in attempt to extend Kasteleyn theory for hyperdeterminants, Lammers [12] expressed number of perfect matchings of a hypergraph as hyperdeterminant of Kasteleyn hypermatrix, by analogy with $d=2$ case. In [13], the author expresses so called Alon-Tarsi number as hyperdeterminant evaluated at Levi-Cevita tensor, i.e. the signed sum over Latin squares, which is realted to famous Alon-Tarsi and Rota-Basis conjectures. Matsumoto shown that hyperdeterminant has nice applications it algebraic combinatorics [21].

In quantum mechanics, hyperdeterminants (in general sense) have applications in characterizing multipartite entanglement. In [17] the authors show, that in some sense, hyperdeterminant shows the measure of entanglement of even number of qubit state.

Hyperdeterminants have applications in proving lower bounds on tensor ranks. In [8] it was demonstrated that nonzero hyperdeterminants imply lower bounds on certain types of tensor ranks. This result applies certain ranks of tensors, providing upper bounds on some generalizations of colored sum-free sets based on constraints related to order polytopes. An appealing feature of hyperdeterminants is that they are explicit and can be a good algebraic tool allowing computations using various operations, which also includes methods for constructing tensors with nonzero hyperdeterminants, see [8].

Main provisions

We denote $[n] = \{0, \dots, n-1\}$. We call the set $[n]^d$ box and refer to elements of the box $[n]^d$ as cells. A slice of $[n]^d$ is a subset of all cells with fixed i-th coordinate (called direction) for some $i \in [d]$. A diagonal of the box $[n]^d$ is a subset of size n with no two cells lying in the same slice. Then the sum (1) of hyperdeterminant runs over diagonals of a box $[n]^d$.

Let $V = (\mathbb{C}^n)^{\otimes d}$ be the space of tensors (state space). Elements of V written in a fixed basis correspond to hypermatrices $\left(X_{i_1,\dots,i_d}\right)$ indexed by $(i_1, \dots, i_d) \in [n_1] \times \dots \times [n_d]$ and we shall identify tensors in V with corresponding hypermatrices by the following:

$$X = \sum_{i_1,\dots,i_d \in [n]} X(i_1, \dots, i_d) e_{i_1} \otimes \dots \otimes e_{i_d}. \quad (2)$$

For given hypermatrix $X \in T_d(n)$ and k-element subsets d-tuple $(I_1, \dots, I_d) \in \binom{[n]}{k}^d$ we denote X_I the subhypermatrix resulting in restricting set of indices in r-th direction to I_r, and hyperdeterminant of X_I we call I-minor.

The group $G = SL(n)^{\times d}$, of d –tuple of matrices of determinant equal to 1, naturally acts on the space of tensors $V = (\mathbb{C}^n)^{\otimes d}$ by

$$(A_1, \dots, A_d) v_1 \otimes \dots \otimes v_d = A_1 v_1 \otimes \dots \otimes A_d v_d \quad (3)$$

for $A_i \in SL(n), v_i \in \mathbb{C}^n$ and extended multilinearly.

Polynomial $P(X)$ with argument $X \in T_d(n)$ is called G-invariant (we will simply write invariant) if $P(g \cdot X) = P(X)$ for any $g \in G$. Then the combinatorial hyperdeterminant (we will simply write hyperdeterminant) is the unique G-invariant polynomial of degree n on tensors $T_d(n)$ [7]. Hyperdeterminant possess the following defining property, similar to the ordinary determinant.

Proposition 2. (Defining properties) Let $F: T_d(n) \to \mathbb{C}$ be the function that satisfy:

(a) (Multilinearity) F is multilinear in slices in each direction, i.e. for first direction

$$F(X + \alpha e_i \otimes Y + \beta e_i \otimes Z) = \alpha F(X + e_i \otimes Y) + \beta F(X + e_i \otimes Y)$$

where X is tensor with i-th slice equal to zero and $Y, Z \in T_{d-1}(n)$, and similar for other directions.

(b) (Skew-symmetry) If two slices in fixed direction of tensor X coincide, then

$$F(X) = 0.$$

(c) (Normalization) $F(I_n) = 1$.

Then $F = DET$.

Materials and Methods

We first introduce the result of Barvinok. For that, we need the following lemma. For simplicity of notation, let us denote the set difference between d-tuples of sets: if $A, B \subseteq [n]^d$ then define $A \backslash\backslash B := A_1 \backslash B_1 \times \ldots \times A_d \backslash B_d$.

Lemma 3. (Laplace Expansion) Let X be a tensor in $T_d(n)$ and $j \in [n]$ fixed. Then

$$DET(X) = \sum_{i=(j,i_2,\ldots,i_d)\in[n]^d} (-1)^{j+i_2+\cdots+i_d-d} X(i) \cdot \mathrm{DET}(\mathrm{X}_{[\mathrm{n}]^{\mathrm{d}}\backslash\backslash i} \quad (4)$$

where $\boldsymbol{i}$ is regarded both as the set $\{i_1\} \times \ldots \times \{i_d\}$ and a cell $(i_1, \ldots, i_d)$.

See [7] for proof of this result. In other words, we expand hyperdeterminant along j-th slice in the first direction, similar to Laplace expansion for ordinary determinant. Here the choice of the first direction

This lemma is crucial in the following algorithms as it provides a recursive formula of calculation of a hyperdeterminant.

A. Barvinok Algorithm

Input: a natural number d and n, d-dimensional tensor X of length n

$$X = \{X(i_1, \ldots, i_d) : 1 \leq i_1, \ldots, \mathrm{i_d} \leq n\}$$

Output: the number $DET(X)$.

Algorithm: Use the dynamic programming based on recurrence (4). Iterate over tuple subsets $I = (I_1, \ldots, I_d) \in \binom{[n]}{k}^d$ in increasing order of $k = 2, \ldots, n$ and compute the I-minor $DET(X_I)$. The base case $k = 1$ is hyperdeterminant of single entry hypermatrix, hence for $I_1 = \{i_1\}, \ldots, I_d = \{i_d\}$ has initial value $DET(X_I) = X(i_1, \ldots, i_d)$. Consequently, assume all size $k - 1$ minors are computed, and we are to compute size k minor I. Let $\phi_c(i)$, for $c \in [d], i \in [k]$, denote the i-th element of I_c. Then by Lemma 3:

$$DET(X_I) = \sum_{i=(1,i_2,\ldots,\mathrm{i_d})\in[k]^d} (-1)^{1+i_2+\cdots+i_d-d} X(\phi(\boldsymbol{i})) \cdot DET(X_{I\backslash\phi(i)}), \quad (4)$$

where $\phi(\boldsymbol{i}) = (\phi_1(i_1), \phi_2(i_2), \ldots, \phi_d(i_d))$.

Time and memory complexity: there are $\sum_{k=1}^{n} \binom{n}{k}^d = O(2^{nd})$ states and for the state of cardinality k we compute the value in $O(k^{d-1})$ time, hence the total time complexity is $O(2^{nd} n^{d-1})$.

B. Improved algorithm

The algorithm essentially optimizes the number of states we need to visit. Since Laplace expansion formula offers expansion of the hyperdeterminant along any slice, we will proceed by calculating minors in the first k slices of first direction, i.e. $I_1 = \{1, \ldots, k\}$ at k-th step.

Input: a natural number d and n, d-dimensional tensor X of length n

$$X = \{X(i_1, \ldots, i_d) : 1 \leq i_1, \ldots, i_d \leq n\}$$

Output: the number $DET(X)$.

Algorithm: iterate over $k = 1$, ...,n and fix $I_1 = \{1, \ldots, k\}$. Iterate over all possible subsets $I_2, \ldots, I_d \subseteq [n]$ such that $|I_c| = k$. As before, assume all minors of size $k - 1$ with set of indices in the first direction I_1 equal to $[k - 1] = \{1, \ldots, k - 1\}$ are computed and we are to compute I-minor.

Let $\phi: I \to [k]^d$ be as above. By Laplace expansion, we expand I-minor of X along the last slice to obtain:

$$DET\left(X_{([k],I_2,\dots,I_k)}\right) \sum_{\mathbf{i}=(k,i_2,\dots,i_d)\in[k^d]} X(\mathbf{i}) \cdot DET\left(X_{[k-1],I_2\setminus\{i_2\},\dots,I_d\setminus\{i_d\}}\right). \quad (5)$$

Since the terms in the sum are size-$(k-1)$ minors of X with indices of the first direction restricted to the set $[k-1]$, the algorithm is correct.

Time complexity: we reduce number of states in 2^n times, hence we require $O(2^{n(d-1)}n^{d-1})$ arithmetic operations and $O(2^{n(d-1)})$ of space.

Results and Discussion

In this section we show compative results of improved algorithm. While the asymtotic behaviour of number of operations involved is still exponential, for fixed d, the improvement provides dramatic increase in performance, allowing to compute hyperdeterminant of a tensor for $n \leq 20$ in reasonable time on a standard computer.

Table 1 – Comparison of performance of original and improved algorithms for d = 4

	Algorithm 1		Algorithm 2	
n	Number of operations	Estimated time	Number of operations	Estimated time
1	8	8 ns	4	4 ns
2	256	256 ns	64	64 ns
3	4,608	4.608 µs	576	576 ns
4	65,536	65.536 µs	4096	4.096 µs
5	819,200	819.200 µs	25600	25.600 µs
6	9,437,184	9.437 ms	147456	147.456 µs
7	102,760,448	102.760 ms	802816	802.816 µs
8	1,073,741,824	1.1 s	4194304	4.2 ms
9	10,871,635,968	11 s	21233664	21 ms
10	107,374,182,400	2 min	104857600	105 ms
11	1,039,382,085,632	17 min	507510784	508 ms
12	9,895,604,649,984	3 hr	2415919104	2.5 s
13	92,908,732,547,072	1 day	11341398016	11 s
14	862,017,116,176,384	10 days	52613349376	53 s
15	7,916,483,719,987,200	92 days	241591910400	4 min
16	72,057,594,037,927,936	2.3 years	1099511627776	18 min
17	650,770,146,155,036,672	20 years	4964982194176	1.4 hr
18	5,836,665,117,072,162,816	186 years	22265110462464	6.2 hr
19	52,025,582,895,383,969,792	1650 years	99230924406784	1.2 days
20	461,168,601,842,738,790,400	14623 years	439804651110400	5 days

See Table 1 for comparison of time performance. The values taken in the table are estimations given that the standard computer performs 10^9 operations per second. Also see Figure 1 that displays the same information graphically.

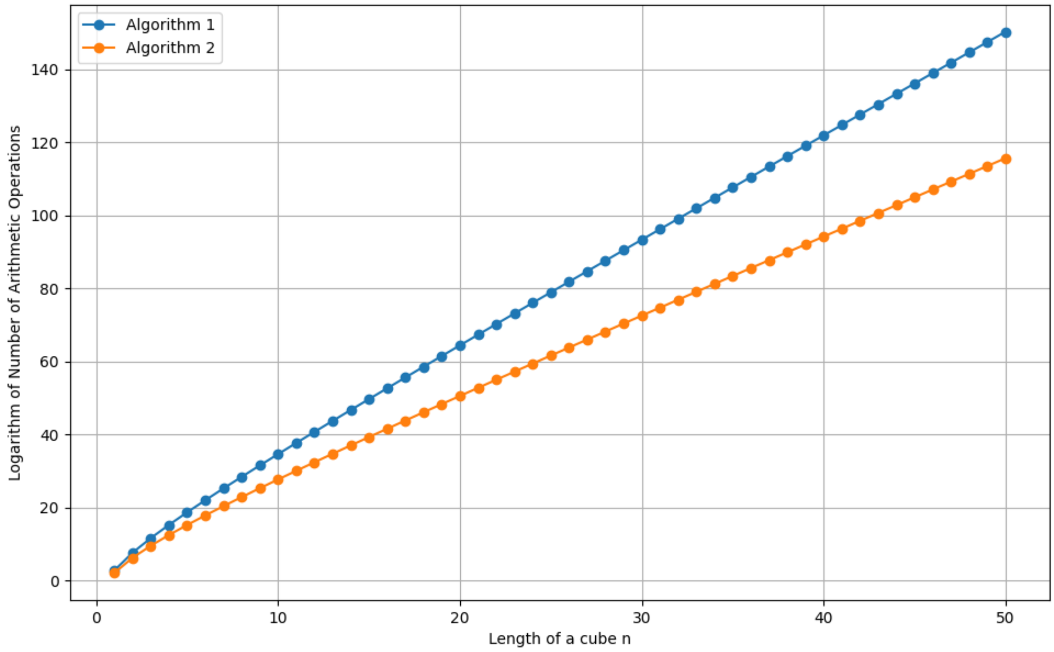

Figure 1 – Comparison of arithmetic operations between algorithms for d = 4

Note that on Figure 1 we have a log base 10 scale of y-axis, so the result looks almost linear.

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REFERENCES

1 Cayley A. On the theory of determinants. Pitt Press, 1844.

2 Cayley A. On the theory of linear transformations. E. Johnson, 1845.

3 Barvinok A.I. New algorithms for linear k-matroid intersection and matroid k-parity problems. Mathematical Programming. 1995 Jul, 69:449–70.

4 Hillar C.J., Lim L.H. Most tensor problems are NP-hard. Journal of the ACM (JACM), 2013, 60(6), 1–39.

5 Gelfand I.M., Kapranov M.M., and Zelevinsky A.V., Hyperdeterminants, Advances in Mathematics, Dec. 1992, vol. 96, no. 2, pp. 226-263. https://doi.org/10.1016/0001-8708(92)90056-q

6 Sokolov N.P. Introduction to the theory of multidimensional matrices, Nukova Dumka, Kiev, 1972. [in Russian]

7 Luque J.G., and Thibon J.Y. Hankel hyperdeterminants and Selberg integrals. Journal of Physics A: mathematical and general, 2003, vol.36, no. 19, p. 5267.

8 Amanov A. and Yeliussizov D. Tensor slice rank and Cayley's first hyperdeterminant. Linear Algebra and its Applications, 2023, 656, pp. 224–246.

9 Cifuentes D. and Parrilo P.A. An efficient tree decomposition method for permanents and mixed discriminants. Linear Algebra and its Applications, 2016, 493, pp. 45–81.

10 Lim L.-H. Tensors and hypermatrices, Handbook of Linear Algebra, 2013, pp. 231–260.

11 Grochow J.A., & Qiao Y. On the Complexity of Isomorphism Problems for Tensors, Groups, and Polynomials I: Tensor Isomorphism-Completeness, Leibniz International Proceedings in Informatics (LIPIcs), 2021, vol. 2021, pp. 31:1-31:19, https://doi.org/10.4230/LIPIcs.ITCS.2021.31.

12 Lammers P.A generalisation of the honeycomb dimer model to higher dimensions. The Annals of Probability, 2021, vol. 49, no. 2, pp. 1033–66.

13 Zappa P. The Cayley determinant of the determinant tensor and the Alon–Tarsi conjecture. Advances in Applied Mathematics, 1997 Jul 1, vol. 19, no.1, pp. 31–44.
14 Blasiak J., Church T., Cohn H., Grochow J.A., Naslund E., Sawin W.F., and C. Umans. On cap sets and the group-theoretic approach to matrix multiplication, Discrete Anal., 2017, paper no. 3, 27 p.
15 Mukhametzhanov B. Half-wormholes in SYK with one time point. SciPost Physics, 2022, vol. 12, no. 1, p. 029.
16 Wigderson A. Non-commutative Optimization-Where Algebra, Analysis and Computational Complexity Meet. InProceedings of the 2022 International Symposium on Symbolic and Algebraic Computation 2022 Jul 4, pp. 13–19.
17 Dobes I., Jing N. Qubits as hypermatrices and entanglement. Physica Scripta. 2024 Apr 11, vol. 99, no. 5, p. 055110.
18 Turatti E. On tensors that are determined by their singular tuples. SIAM Journal on Applied Algebra and Geometry, 2022, vol. 6, no. 2, pp. 319–338.
19 Lim LH. Tensors in computations. Acta Numerica. 2021 May, no. 30, pp. 555–764.
20 Zheng YN. The characteristic polynomial of the complete 3-uniform hypergraph. Linear Algebra and its Applications. 2021 Oct 15, no. 627, pp. 275–286.
21 Matsumoto S. Hyperdeterminantal expressions for Jack functions of rectangular shapes. Journal of Algebra. 2008 Jul 15, vol. 320, no. 2, pp. 612–632.

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КЭЙЛИ БІРІНШІ ГИПЕРДЕТЕРМИНАНТЫН ЕСЕПТЕУДІҢ ЖЕТІЛДІРІЛГЕН АЛГОРИТМІ

Аңдатпа

Комбинаторлық гипердетерминант DET – бұл жұп индекстер саны бар гиперматрица жазбалары бойынша біртекті көпмүше, сондай-ақ ол ең төменгі дәрежелі жалғыз SL-инварианты. Бұл тұжырымды алғаш рет 19 ғасырдың ортасында Кэйли зерттеген. Оның іргелі сипатына байланысты бұл көпмүшенің есебін шығару маңызды мәселеге айналды. Барвинок белгілі бір d және ұзындығы n болатын кубтық гиперматрица X үшін гипердетерминантты есептеудің $O(2^{nd}n^{d-1})$ алгоритмін ұсынды. DET(X) гипердетерминантының берілген гиперматрица X үшін нөлге тең болу мәселесі NP-қатерлі болғандықтан, гипердетерминантты тиімді есептеу алгоритмін анықтау қажет, себебі есептің көлемі экспоненциалды түрде өседі. Біз гипердетерминантты есептеудің жақсартылған алгоритмін ұсынамыз, ол $O(2^{n(d-1)}n^{d-1})$ арифметикалық операцияларды қажет етеді.

Тірек сөздер: инвариантты көпмүшелер, кванттық түйісу, SLOCC.

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УЛУЧШЕННЫЙ АЛГОРИТМ ВЫЧИСЛЕНИЯ ПЕРВОГО ГИПЕРДЕТЕРМИНАНТА КЭЛИ

Аннотация

Комбинаторный гипердетерминант DET – это однородный многочлен от элементов гиперматрицы с четным числом индексов, который является единственным SL-инвариантом минимальной степени, который впервые стал изучать Кэли в середине XIX века. Учитывая его фундаментальную природу, вычисление этого многочлена является важной задачей в разных разделах науки. Для фиксированного d и кубической гиперматрицы X Барвинок предложил алгоритм вычисления гипердетерминанта, используя $\mathrm{O}(2^{\mathrm{nd}}\mathrm{n}^{\mathrm{d-1}})$ арифметических операций. Поскольку задача определения, равен ли гипердетерминант $\mathrm{DET(X)}$ данной гиперматрицы X нулю, является NP-трудной, крайне важно разработать наиболее эффективный алгоритм, так как размер задачи растет экспоненциально. Мы предлагаем улучшенный алгоритм вычисления гипердетерминанта, который требует $\mathrm{O}(2^{\mathrm{n(d-1)}}\mathrm{n}^{\mathrm{d-1}})$ арифметических операций.

Ключевые слова: первый гипердетерминант Кэли, SL-инвариант, разложение Лапласа.

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