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PRE-CONDITIONING METHOD FOR SUBSTANTIALLY SUBSONIC FLOWS

Abstract

The proposed work is a detailed numerical simulation of three-dimensional subsonic turbulent flow in a channel, with the main focus on symmetric perpendicular jets arising from the walls. The solution of the Favre-averaged Navier-Stokes equations closed by the turbulence model is carried out using an algorithm based on the ENO scheme. To accelerate the convergence of the iterative process, a preconditioning method is used and a transition to a vector of primitive variables is performed. The results of the study are essential for a better understanding of subsonic turbulent flows and may find applications in various fields, including engineering and scientific research. The relevance of the work is highlighted by the development of efficient numerical algorithms capable of solving subsonic three-dimensional Navier-Stokes equations using high-order accuracy schemes, as well as the application of robust turbulence models to analyze supersonic multicomponent flow. The scientific novelty of the work is the successful use of the preconditioning method to accelerate the convergence of the iterative process.

Key words: numerical modelling, subsonic flow, perfect gas, boundary layer, Navier-Stokes equations.

Introduction

One of the peculiarities of modelling low-speed flows using the compressible form of the Euler or Navier-Stokes equations is the instability of the numerical solution and the slowing down of the convergence rate of the iterative process due to the small difference between the velocities of acoustic and convective waves [1-4]. The application of preconditioning allows us to modify the difference equations in such a way that the eigenvalues of the Jacobian (wave propagation velocity) of the modified system of equations have comparable orders of magnitude. The use of the preconditioning method [5-14] mainly leads to the improvement of convergence of stationary solutions of the establishment schemes.

When approximating spatial derivatives, central-difference methods or, as in the case of the scheme in [3], methods with special approximations are usually used. The main objective of this work is to develop a numerical algorithm for solving the problem of blowing subsonic jets from circular holes located symmetrically on the upper and lower walls of the channel perpendicular to the flow of a low-velocity perfect gas using the preconditioning method. The solution of the original Favre averaged Navier-Stokes equations is performed using an algorithm based on the ENO scheme.

Based on the analysis, it can be noted that most authors studying flows at low Mach numbers and using the preconditioning procedure to solve the equations use Turkel matrices [7], which are suitable for preconditioning both the Euler and Navier-Stokes equations

The analysis of methods for solving the Navier-Stokes equations for flows with small Mach numbers shows that the most common way of eliminating computational difficulties is the application of the preconditioning method, which allows to solve in a unified way the problems characterized by the velocity change, mainly leads to acceleration of convergence of the stationary solution of the establishment schemes.

Main Provisions

The initial system is a system of three-dimensional Favre averaged Navier-Stokes equations for a compressible turbulent gas written in Cartesian coordinate system in conservative form:

$$\frac{\partial U}{\partial t}+\frac{\partial(E-E_v)}{\partial x}+\frac{\partial(F-F_v)}{\partial y}+\frac{\partial(G-G_v)}{\partial z}=0 \quad (1)$$

$$U=(\rho,\rho u,\rho v,\rho w,E_t)^T \quad (2)$$

Expressions for convective E, F, G and diffusive E_v, F_v, G_v flux vectors are given in [15]. The initial system (1) is written in dimensionless form. The input parameters are taken as defining parameters $u_\infty, \rho_\infty, T_\infty$ pressure and total energy are related to the value of $\rho_\infty u_\infty^2$, the characteristic length dimension is the diameter of the jet's circular orifice.

The flow parameters are given at the inlet and also as initial data:

$u=1, v=0, w=0, \rho=1, T=1 \qquad x=0, 0\le y\le H_y, 0\le z\le H_z$

Initial data for the parameters k, ω are determined on the assumption of equality of turbulence generation and its dissipation

$$P_k=\beta^*\rho\omega k$$

Then

$k=k_\infty$, where $k_\infty=\frac{\mu_{tB-L}}{\rho Re\sqrt{\beta^*}}\sqrt{\frac{P_k}{\mu_{tB-L}}}$, $\omega=\omega_\infty$,

where $\omega_\infty=\frac{\rho k}{\mu_{tB-L}\,Re}$, $P_k=\frac{\mu_t}{Re}\left(\left(\frac{\partial w}{\partial x}\right)^2+\frac{4}{3}\left(\frac{\partial w}{\partial z}\right)^2\right)$.

Here, the algebraic Baldwin-Lomax model is used to determine the value of the turbulent viscosity coefficient. Near the wall ($0\le z\le\delta_1$), the turbulent viscosity coefficient has the following form

$$\mu_t=\rho l^2|\Omega|,$$

where $|\Omega|$ – vorticity, $l=kz\left|1-e^{-z^+/A}\right|$ – mixing path length, $k=0.41$ – Karman constant, A$=26$. Away from the wall ($z>\delta_1$) is accepted

$$\mu_t = 0.0168\rho V_0 L_0,$$

where $V_0 = (F_{max}, 0.25q_{dif}^2/F_{max})$, $L_0 = 1.6z_{max}I^k$, $F_{max} = max(|\Omega|lk)$, z_{max} consistent with F_{max}, $q_{dif} = max(|\vec{V}|) - min(||\vec{V}||)$, $\vec{V} = \sqrt{u^2 + v^2 + w^2}$, $I^k = [1 + 5.5(0.3z/z_{max})^6]^{-1}$ – Klebanov's limiting multiplier [16]. This is the time during which turbulent flow is observed at a given point, related to the total measurement time. In almost all algebraic turbulence models, the Klebanov intermittency coefficient is used as a multiplier to the turbulent viscosity in the outer region.

on the bottom wall:

$$u = 0, v = 0, w = 0, \frac{\partial T}{\partial z} = 0, \frac{\partial P}{\partial z} = 0, z = 0, 0 \le x \le H_x, 0 \le y \le H_y.$$

The following boundary conditions are specified for the parameters $k - \omega$ of the turbulence model on the wall

$$k = 0, \omega = \frac{6\mu}{0.075\rho(\Delta y_1)^2}.$$

There is a boundary layer near the wall, the thickness of which is determined by the formula $\delta_1 = 0.37x(Re)^{-0.2}$. A wall layer is also specified (10% of the boundary layer) $\delta_2 = 0.1\delta_1$. In this situation, the use of the percent boundary layer thickness does not lead to significant difficulties, since the calculation results are not sensitive to this parameter. Longitudinal velocity component u takes the following form:

$$u = 0{,}1\left(\frac{z}{\delta_2}\right) + 0{,}9\left(\frac{z}{\delta_2}\right)^2, \qquad 0 \le x \le H_x, 0 \le y \le H_y, \ 0 \le z \le \delta_2$$

In a developed turbulent boundary layer, the longitudinal velocity profile is given by a power law:

$$u = \left(\frac{z}{\delta_1}\right)^{1/7}, \qquad 0 \le x \le H_x, 0 \le y \le H_y, \delta_1 \le z \le \delta_2.$$

Depending on the velocity distribution, the temperature and density values will take the form of:

$$T = T_w + u(1 - T_w), \rho = \frac{1}{T},$$

где $T_w = \left(1 + r\frac{(\gamma-1)}{2}M_\infty^2\right)$ – temperature on the wall, $r = 0.88$;

on the stream:

$$u = 0, v = 0, T = 0.6, w = \sqrt{T}M_0/M_\infty, P_0 = nP_\infty, \qquad z = 0, M_\infty|x^2 + y^2| \le R;$$

the symmetry condition is set on the upper boundary:

$$w = 0, \frac{\partial u}{\partial z} = 0, \frac{\partial v}{\partial z} = 0, \frac{\partial T}{\partial z} = 0, \frac{\partial k}{\partial z} = 0, \frac{\partial \omega}{\partial z} = 0, z = H_z, 0 \le x \le H_x, 0 \le y \le H_y$$

on the lateral borders:

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial y} = \frac{\partial \rho}{\partial y} = \frac{\partial k}{\partial y} = \frac{\partial \omega}{\partial y} = 0 \qquad y = 0,\ 0 \le x \le H_x,\ 0 \le z \le H_z$$

where H_x - length, H_z - height, H_y - design area width, R - radius of round hole; the non-reflection condition is set on the output boundary.

Materials and methods

In [15], an ENO scheme is constructed and the applicability of the scheme to the solution of the problem of supersonic flow of a multicomponent gas in a channel with blowing perpendicular jets is shown. In accordance with this, mesh densification is introduced in the boundary layer, near the wall and at the jet level, with the help of transformations [15] for a more accurate account of the flow:

$$\xi = \xi(x),\ \eta = \eta(z),\ \zeta = \zeta(y). \tag{3}$$

In this case the system of equations (1) in generalised coordinates is written in the form:

$$\frac{\partial \tilde{U}}{\partial t} + \frac{\partial \tilde{E}}{\partial \xi} + \frac{\partial \tilde{F}}{\partial \eta} + \frac{\partial \tilde{G}}{\partial \zeta} = \frac{\partial \tilde{E}_{v2}}{\partial \xi} + \frac{\partial \tilde{E}_{vm}}{\partial \xi} + \frac{\partial \tilde{F}_{v2}}{\partial \eta} + \frac{\partial \tilde{F}_{vm}}{\partial \eta} + \frac{\partial \tilde{G}_{v2}}{\partial \zeta} + \frac{\partial \tilde{G}_{vm}}{\partial \zeta} \tag{4}$$

where $\tilde{U} = \frac{1}{J}\vec{U},\ \tilde{E} = \left(\frac{\xi_x}{J}\right)\vec{E},\ \tilde{F} = \left(\frac{\eta_z}{J}\right)\vec{F},\ \tilde{E}_{v2} = \left(\frac{\xi_x}{J}\right)\vec{E}_{v2},\ \tilde{E}_{vm} = \left(\frac{\xi_x}{J}\right)\vec{E}_{vm},$

$\tilde{F}_{v2} = \left(\frac{\eta_z}{J}\right)\vec{F}_{v2},\ \tilde{F}_{vm} = \left(\frac{\eta_z}{J}\right)\vec{F}_{vm},\ \tilde{G}_{v2} = \left(\frac{\zeta_y}{J}\right)\vec{G}_{v2},\ \tilde{G}_{vm} = \left(\frac{\zeta_y}{J}\right)\vec{G}_{vm},$

where $J = \frac{\partial(\xi,\eta,\zeta)}{\partial(x,y,z)}$ - Jacobian transformation, $\tilde{E}_{vm}, \tilde{E}_{v2},$ – diffusion terms containing mixed and second derivatives.

In order to eliminate the problem associated with the system of equations (1), where $M \to 0$ the method of preconditioning is used. For this purpose, let us consider the system of equations for compressible fluid flow in linearised form in three-dimensional formulation

$$\frac{\partial Q}{\partial t} + A\frac{\partial Q}{\partial x} + B\frac{\partial Q}{\partial y} + C\frac{\partial Q}{\partial z} = 0 \tag{5}$$

where $A = \frac{\partial E}{\partial x}, B = \frac{\partial F}{\partial y}, C = \frac{\partial G}{\partial z}$ – Jacobi matrix, and the vector of primitive variables has the form $Q = (\rho, u, v, w, S)$.

The Jacobi matrices are defined as:

$$A = \begin{pmatrix} u & \rho c^2 & 0 & 0 & 0 \\ \frac{1}{\rho} & u & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & u \end{pmatrix} \quad B = \begin{pmatrix} v & 0 & \rho c^2 & 0 & 0 \\ 0 & v & 0 & 0 & 0 \\ \frac{1}{\rho} & 0 & v & 0 & 0 \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & v \end{pmatrix} \quad C = \begin{pmatrix} w & 0 & 0 & \rho c^2 & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ \frac{1}{\rho} & 0 & 0 & w & 0 \\ 0 & 0 & 0 & 0 & w \end{pmatrix}$$

The idea of the method is to modify the system of equations (5) by multiplying the terms with time derivative by the matrix P^{-1}:

$$P^{-1}\frac{\partial Q}{\partial t}+A\frac{\partial Q}{\partial x}+B\frac{\partial Q}{\partial y}+C\frac{\partial Q}{\partial z}=0 \tag{6}$$

multiplied by P^{-1}, the system of equations (6) is rewritten in the form:

$$\frac{\partial Q}{\partial t}+PA\frac{\partial Q}{\partial x}+PB\frac{\partial Q}{\partial y}+PC\frac{\partial Q}{\partial z}=0 \tag{7}$$

The main purpose of building a preconditioning matrix P^{-1} s to make the eigenvalues of the matrices *PA* and *PB*, *PC* have the same order of magnitude. It is assumed that the matrix P^{-1} positive and as vector parameters different independent variables can be used (*p, u, v, w, P, T, S)*. Selection of primitive vector parameters Q is dictated only by convenience in the construction of preconditioning matrices. Turkel's preconditioning matrix [7] was used in this work:

$$P^{-1}=\begin{pmatrix} \frac{c^2}{\beta^2} & 0 & 0 & 0 & \delta \\ \frac{\alpha u}{\rho\beta^2} & 1 & 0 & 0 & 0 \\ \frac{\alpha v}{\rho\beta^2} & 0 & 1 & 0 & 0 \\ \frac{\alpha w}{\rho\beta^2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and correspondingly its inverse matrix:

$$P=\begin{pmatrix} \frac{\beta^2}{c^2} & 0 & 0 & 0 & -\frac{\beta^2}{c^2}\delta \\ -\frac{\alpha u}{\rho c^2} & 1 & 0 & 0 & \frac{\alpha u}{\rho c^2}\delta \\ -\frac{\alpha v}{\rho c^2} & 0 & 1 & 0 & \frac{\alpha v}{\rho c^2}\delta \\ -\frac{\alpha w}{\rho c^2} & 0 & 0 & 1 & \frac{\alpha w}{\rho c^2}\delta \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

constructed for the vector of independent variables, $Q=(p,u,v,w,S)^T$, here $S=h\left(p/\tilde{n}^{\tilde{a}}\right)$ – entropy. Then products of matrices *PA* и *PB*, *PC* will take the form

$$PA=\begin{pmatrix} \beta^2 u & \rho\beta^2 & 0 & 0 & -\frac{\beta^2 u\delta}{c^2} \\ \frac{1}{\rho}\left(1-\frac{\alpha u^2}{c^2}\right) & (1-\alpha)u & 0 & 0 & \frac{\alpha u^2\delta}{\rho c^2} \\ -\frac{\alpha vu}{\rho c^2} & -\alpha v & u & 0 & \frac{\alpha vu\delta}{\rho c^2} \\ -\frac{\alpha wu}{\rho c^2} & -\alpha w & 0 & u & \frac{\alpha wu\delta}{\rho c^2} \\ 0 & 0 & 0 & 0 & u \end{pmatrix}$$

$$PB=\begin{pmatrix} \frac{\beta^2 v}{c^2} & 0 & \rho\beta^2 & 0 & -\frac{\beta^2 v\delta}{c^2} \\ -\frac{\alpha vu}{\rho c^2} & v & -\alpha u & 0 & \frac{\alpha vu\delta}{\rho c^2} \\ \frac{1}{\rho}\left(1-\frac{\alpha v^2}{c^2}\right) & 0 & (1-\alpha)v & 0 & \frac{\alpha v^2\delta}{\rho c^2} \\ -\frac{\alpha wv}{\rho c^2} & 0 & -\alpha w & v & \frac{\alpha wv\delta}{\rho c^2} \\ 0 & 0 & 0 & 0 & v \end{pmatrix}$$

$$PC=\begin{pmatrix} \frac{\beta^2 w}{c^2} & 0 & 0 & \rho\beta^2 & -\frac{\beta^2 w\delta}{c^2} \\ -\frac{\alpha uw}{\rho c^2} & w & 0 & -\alpha u & \frac{\alpha wu\delta}{\rho c^2} \\ -\frac{\alpha vw}{\rho c^2} & 0 & w & -\alpha v & \frac{\alpha vw\delta}{\rho c^2} \\ \frac{1}{\rho}\left(1-\frac{\alpha w^2}{c^2}\right) & 0 & 0 & (1-\alpha)w & \frac{\alpha w^2\delta}{\rho c^2} \\ 0 & 0 & 0 & 0 & w \end{pmatrix}$$

Let us calculate the eigenvalues of the matrix PA. For this purpose we solve the system of homogeneous equations of the form: $(PA - \Lambda I) = 0$ with respect to the unknown variables $\lambda_i \quad i=1...4$.

$$\lambda = u,$$

$$\lambda_{\pm} = -\frac{1}{2c^2}\left[zu \mp \sqrt{z^2u^2 + 4\beta^2 - \frac{u^2}{c^2}}\right],$$

where $z = 1-\alpha+\frac{\beta^2}{c^2}$ (if $\alpha = 1+\frac{\beta^2}{c^2}$, when z = 0, то $\lambda_{\pm} = \mp\sqrt{\beta^2\left(1-M^2\right)}$).

Thus, as a result of preconditioning in the case of M<<1, in selecting $\beta \approx u^2+v^2+w^2$ all eigenvalues of the Jacobi matrix *PA* will have the same order u. Moreover or $\beta = c^2$ all λ_i become eigenvalues of the original Jacobi matrix A.

The eigenvalues of the matrices *PB* and *PC* are determined similarly. Then, according to the calculated eigenvalues of the matrices, the eigenvectors from the solution of the following system of homogeneous linear equations are determined:

$$\left(PA\cdot W_i - \Lambda W_i\right) = 0,$$

where $W_i \quad i=1...5$ – components of eigenvectors corresponding to the eigenvalue $\lambda_i \quad i=1...5$. The matrices of eigenvectors (right eigenvectors) will take the form:

$$R_x = \begin{pmatrix} \rho\beta^2 & \rho\beta^2 & 0 & 0 & 0 \\ \lambda_+ - \dfrac{\beta^2 u}{c^2} & \lambda_- - \dfrac{\beta^2 u}{c^2} & 0 & 0 & \dfrac{\delta u}{\rho c^2} \\ \dfrac{\alpha v \lambda_+}{u-\lambda_+} & \dfrac{\alpha v \lambda_-}{u-\lambda_-} & 1 & 0 & 0 \\ \dfrac{\alpha w \lambda_+}{u-\lambda_+} & \dfrac{\alpha w \lambda_-}{u-\lambda_-} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_y = \begin{pmatrix} \rho\beta^2 & 0 & \rho\beta^2 & 0 & 0 \\ \dfrac{\alpha u \lambda_+}{v-\lambda_+} & 1 & \dfrac{\alpha u \lambda_-}{v-\lambda_-} & 0 & 0 \\ \lambda_+ - \dfrac{\beta^2 v}{c^2} & 0 & \lambda_- - \dfrac{\beta^2 v}{c^2} & 0 & \dfrac{\delta v}{\rho c^2} \\ \dfrac{\alpha w \lambda_+}{v-\lambda_+} & 0 & \dfrac{\alpha w \lambda_-}{v-\lambda_-} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_z = \begin{pmatrix} \beta^2 & 0 & 0 & \beta^2 & 0 \\ \dfrac{\alpha u \lambda_+}{w-\lambda_+} & 1 & 0 & \dfrac{\alpha u \lambda_-}{w-\lambda_-} & 0 \\ \dfrac{\alpha v \lambda_+}{w-\lambda_+} & 0 & 1 & \dfrac{\alpha v \lambda_-}{w-\lambda_-} & 0 \\ \lambda_+ - \dfrac{\beta^2 w}{c^2} & 0 & 0 & \lambda_- - \dfrac{\beta^2 w}{c^2} & \dfrac{\delta w}{\rho c^2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The inverse matrices or left eigenvectors will be written in the following form:

$$R_x^{-1} = \begin{pmatrix} \dfrac{-\gamma c^2 - \beta^2 u}{c^2 \beta\rho(-\gamma+\lambda)} & \dfrac{1}{-\gamma+\lambda} & 0 & 0 & -\dfrac{\delta u}{c^2\rho(-\gamma+\lambda)} \\ -\dfrac{-\lambda c^2 + \beta^2 u}{c^2\beta^2\rho(-\gamma+\lambda)} & -\dfrac{1}{-\gamma+\lambda} & 0 & 0 & \dfrac{\delta u}{c^2\rho(-\gamma+\lambda)} \\ -\dfrac{(-\gamma c^2\lambda + \beta^2 u^2)\alpha v}{c^2\beta^2\rho(u-\gamma)(u-\lambda)} & -\dfrac{\alpha v u}{(u-\gamma)(u-\lambda)} & 1 & 0 & \dfrac{\delta\alpha v u^2}{c^2\rho(u-\gamma)(u-\lambda)} \\ -\dfrac{(-\gamma c^2\lambda + \beta^2 u^2)\alpha w}{c^2\beta^2\rho(u-\gamma)(u-\lambda)} & -\dfrac{\alpha w u}{(u-\gamma)(u-\lambda)} & 0 & 1 & \dfrac{\delta\alpha w u^2}{c^2\rho(u-\gamma)(u-\lambda)} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_y^{-1}=\begin{pmatrix} \frac{-\gamma c^2+\beta^2 v}{c^2\beta^2\rho(-\gamma+\lambda)} & 0 & \frac{1}{-\gamma+\lambda}0 & 0 & -\frac{\delta v}{c^2\rho\left(-\gamma+\lambda\right)} \\ -\frac{\left(-\gamma c^2+\beta^2 v^2\right)\alpha u}{c^2\beta^2\rho(v-\gamma)(v-\lambda)} & 0 & -\frac{\alpha v u}{(v-\gamma)\left(v-\lambda\right)} & 0 & \frac{\delta\alpha u v^2}{c^2\rho(v-\gamma)\left(v-\lambda\right)} \\ -\frac{-\lambda c^2+\beta^2 v}{c^2\beta^2\rho(-\gamma+\lambda)} & 1 & -\frac{1}{-\gamma+\lambda} & 0 & \frac{\delta v}{c^2\rho\left(-\gamma+\lambda\right)} \\ -\frac{\left(-\gamma c^2\lambda+\beta^2 v^2\right)\alpha w}{c^2\beta^2\rho(v-\gamma)(v-\lambda)} & 0 & -\frac{\alpha w\vartheta}{(v-\gamma)\left(v-\lambda\right)}0 & 1 & \frac{\delta\alpha w v^2}{c^2\rho(v-\gamma)\left(v-\lambda\right)} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_z^{-1}=\begin{pmatrix} \frac{-\gamma c^2+\beta^2 w}{c^2\beta^2\rho(-\gamma+\lambda)} & 0 & 0 & \frac{1}{-\gamma+\lambda} & -\frac{\delta w}{c^2\rho\left(-\gamma+\lambda\right)} \\ -\frac{\left(-\gamma c^2\lambda+\beta^2 w^2\right)\alpha u}{c^2\beta^2\rho(w-\gamma)(w-\lambda)} & 1 & 0 & -\frac{\alpha u w}{(w-\gamma)\left(w-\lambda\right)} & \frac{\delta\alpha u w^2}{c^2\rho(w-\gamma)\left(w-\lambda\right)} \\ -\frac{\left(-\gamma c^2\lambda+\beta^2 w^2\right)\alpha v}{c^2\beta^2\rho(w-\gamma)(w-\lambda)} & 0 & 1 & -\frac{\alpha v w}{(w-\gamma)\left(w-\lambda\right)} & \frac{\delta\alpha v w^2}{c^2\rho(w-\gamma)\left(w-\lambda\right)} \\ -\frac{-\lambda c^2+\beta^2 w}{c^2\beta^2\rho(-\gamma+\lambda)} & 0 & 0 & -\frac{1}{\gamma-\lambda} & \frac{\delta w}{c^2\rho\left(-\gamma+\lambda\right)} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The following relations must be satisfied for the constructed matrices:

$$R_x \times R_x^{-1}=I,\quad R_y \times R_y^{-1}=I,\quad R_z \times R_z^{-1}=I \quad \text{– unit matrices.}$$

$$R_x\Lambda_x R_x^{-1}=PA,\quad R_y\Lambda_y R_y^{-1}=PB,\quad R_z\Lambda_z R_z^{-1}=PC. \tag{8}$$

Let us make a transition from the system of equations (5) in primitive variables to the system of equations (1) in conservative variables. Then equation (1) is rewritten in the form:

$$\frac{\partial U}{\partial t}=\frac{\partial Q}{\partial U}PA\frac{\partial U}{\partial Q}\frac{\partial U}{\partial x}+\frac{\partial Q}{\partial U}PB\frac{\partial U}{\partial Q}\frac{\partial U}{\partial y}+\frac{\partial Q}{\partial U}PC\frac{\partial U}{\partial Q}\frac{\partial U}{\partial z} \tag{9}$$

Moreover, the transition matrix has the form:

$$\frac{\partial Q}{\partial U}=\begin{pmatrix} \frac{1}{c^2} & 0 & 0 & 0 & -\frac{1}{c^2} \\ \frac{u}{c^2} & \rho & 0 & 0 & -\frac{u}{c^2} \\ \frac{v}{c^2} & 0 & \rho & 0 & -\frac{v}{c^2} \\ \frac{w}{c^2} & 0 & 0 & \rho & -\frac{w}{c^2} \\ \frac{1}{\gamma-1}+\frac{M^2}{2} & \rho u & \rho v & \rho w & \frac{M^2}{2} \end{pmatrix}$$

The inverse of it will be written as follows:

The matrix of preconditioning in conservative variables in the x direction taking into account (5) is as follows:

According to the principle of ENO scheme construction, the original system of equations is formally represented as follows:

$$\frac{\partial U}{\partial Q}=\begin{pmatrix} \frac{\gamma-1}{2}\left(u^2+v^2+w^2\right) & -(\gamma-1)u & -(\gamma-1)v & -(\gamma-1)w & \gamma-1 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 \\ -\frac{\vartheta}{\rho} & 0 & \frac{1}{\rho} & 0 & 0 \\ -\frac{w}{\rho} & 0 & 0 & \frac{1}{\rho} & 0 \\ \frac{\gamma-1}{2}\left(u^2+v^2+w^2\right)-c^2 & -(\gamma-1)u & -(\gamma-1)v & -(\gamma-1)w & \gamma-1 \end{pmatrix}$$

The matrix of preconditioning in conservative variables in the x direction taking into account (5) is as follows:

$$\frac{\partial Q}{\partial U}PA\frac{\partial U}{\partial Q}=\frac{\partial Q}{\partial U}R_x^{-1}\Lambda_x R_x\frac{\partial U}{\partial Q} \qquad (10)$$

According to the principle of ENO scheme construction, the original system of equations is formally represented as follows:

$$\frac{\partial \tilde{U}}{\partial t}+\left(\hat{A}^{+}+\hat{A}^{-}\right)A_{pr}\frac{\partial \tilde{U}}{\partial \xi}+\left(\hat{B}^{+}+\hat{B}^{-}\right)B_{pr}\frac{\partial \tilde{U}}{\partial \eta}+\left(\hat{C}^{+}+\hat{C}^{-}\right)C_{pr}\frac{\partial \tilde{U}}{\partial \zeta}=$$

$$=\left[\frac{\partial\left(\tilde{E}_{v2}+\tilde{E}_{vm}\right)}{\partial \xi}+\frac{\partial\left(\tilde{F}_{v2}+\tilde{F}_{vm}\right)}{\partial \eta}+\frac{\partial\left(\tilde{G}_{v2}+\tilde{G}_{vm}\right)}{\partial \zeta}\right]-$$

$$\left(\hat{A}^{+}+\hat{A}^{-}\right)\frac{\partial(\vec{E}_\xi+\vec{D}_\xi)}{\partial \xi}-\left(\hat{B}^{+}+\hat{B}^{-}\right)\frac{\partial(\vec{E}_\eta+\vec{D}_\eta)}{\partial \eta}-\left(\hat{C}^{+}+\hat{C}^{-}\right)\frac{\partial(\vec{E}_\zeta+\vec{D}_\zeta)}{\partial \zeta}$$

here $A_{pr}=\frac{\partial Q}{\partial U}PA\frac{\partial U}{\partial Q}$ $B_{pr}=\frac{\partial Q}{\partial U}PB\frac{\partial U}{\partial Q}$ $C_{pr}=\frac{\partial Q}{\partial U}PC\frac{\partial U}{\partial Q}$ – Jacobi matrices,

$\hat{A}^{\pm}=R_x\hat{\Lambda}_\xi R_x^{-1}=R_x\left(\frac{1\pm sign(\Lambda_\xi)}{2}\right)R_x^{-1}$ $\hat{B}^{\pm}=R_y\hat{\Lambda}_\eta R_y^{-1}=R_y\left(\frac{1\pm sign(\Lambda_\eta)}{2}\right)R_y^{-1}$

$\hat{C}^{\pm}=R_z\hat{\Lambda}_\zeta R_z^{-1}=R_z\left(\frac{1\pm sign(\Lambda_\zeta)}{2}\right)R_z^{-1}$ $E^m=\tilde{E}+\vec{E}_\xi+\vec{D}_\xi$ $F^m=\tilde{F}+\vec{E}_\eta+\vec{D}_\eta$

$G^m=\tilde{G}+\vec{E}_\zeta+\vec{D}_\zeta$

modified flows at nodal points (i,j.k), consisting of initial convective vectors $(\tilde{E},\tilde{F},\tilde{G})$ and high-order precision additive terms $(\vec{E}_\xi,\vec{D}_\xi,\vec{E}_\eta,\vec{D}_\eta,\vec{E}_\zeta,\vec{D}_\zeta)$, described in detail in [17].

After factorisation of the one-step finite-difference scheme for time integration of equation (11), the following equality is obtained:

$$\left\{I + \Delta t\left[\left(\hat{A}^{+} + \hat{A}^{-}\right)^{n}\frac{\partial}{\partial \xi}A_{\xi}^{n}\cdot - \frac{\partial}{\partial \xi}\tilde{\mu}_{\xi}\frac{\partial}{\partial \xi}\frac{1}{\tilde{U}_{1}}\cdot\right]\right\}\times$$
$$\left\{I + \Delta t\left[\left(\hat{B}^{+} + \hat{B}^{-}\right)^{n}\frac{\partial}{\partial \eta}B_{\eta}^{n}\cdot - \frac{\partial}{\partial \eta}\tilde{\mu}_{\eta}\frac{\partial}{\partial \eta}\frac{1}{\tilde{U}_{1}}\cdot\right]\right\}\times$$
$$\times\left\{I + \Delta t\left[\left(\hat{Q}^{+} + \hat{Q}^{-}\right)^{n}\frac{\partial}{\partial \zeta}Q_{\zeta}^{n}\cdot - \frac{\partial}{\partial \zeta}\tilde{\mu}_{\zeta}\frac{\partial}{\partial \zeta}\frac{1}{\tilde{U}_{1}}\cdot\right]\right\}\tilde{U}^{n+1} =$$
$$= \tilde{U}^{n} + \Delta t\left[\frac{\partial \tilde{E}_{v2}^{n}}{\partial \xi} + \frac{\partial \tilde{F}_{v2}^{n}}{\partial \eta} + \frac{\partial \tilde{G}_{v2}^{n}}{\partial \zeta} + \frac{\partial}{\partial \xi}\left(2\tilde{E}_{vm}^{n} - \tilde{E}_{vm}^{n-1}\right) + \frac{\partial}{\partial \eta}\left(2\tilde{F}_{vm}^{n} - \tilde{F}_{vm}^{n-1}\right) + \frac{\partial}{\partial \zeta}\left(2\tilde{G}_{vm}^{n} - \tilde{G}_{vm}^{n-1}\right)\right] - \Delta t\left[\left(\hat{A}^{+} + \hat{A}^{-}\right)\frac{\partial}{\partial \xi}\left(E_{\xi} + D_{\xi}\right) + \left(\hat{B}^{+} + \hat{B}^{-}\right)\frac{\partial}{\partial \eta}\left(E_{\eta} + D_{\eta}\right) + \left(\hat{Q}^{+} + \hat{Q}^{-}\right)\frac{\partial}{\partial \zeta}\left(E_{\zeta} + D_{\zeta}\right)\right]^{n} \quad (12)$$

where $A_{\xi} = \xi_x A, B_{\eta} = \eta_z B, Q_{\zeta} = \zeta_y Q$, moreover $\hat{A}^{+} + \hat{A}^{-} = I, I$ - unit matrix $\tilde{\mu}_{\xi} = \frac{\mu \xi_x^2}{Re\, J}, \tilde{\mu}_{\eta} = \frac{\mu \eta_z^2}{Re\, J}, \tilde{\mu}_{\zeta} = \frac{\mu \zeta_y^2}{Re\, J}$.

The following operator is used to approximate the derivatives in the convective terms:

$$\left(\hat{A}^{-} + \hat{A}^{+}\right)\frac{\partial}{\partial \xi}f|_{ij} = \frac{\hat{A}_{i+1/2}^{-}\left(f_{i+1j} - f_{ij}\right) + \hat{A}_{i-1/2}^{+}\left(f_{ij} - f_{i-1j}\right)}{\Delta \xi}$$

The approximation of the terms containing additive vectors of high order is performed as in [18].

Results and discussion

A splitting method is applied to solve the system of equations (11), using matrix run vector. A turbulence model is introduced to equilibrate the system of equations (1), and a precondition method $k - \omega$ is implemented to overcome the stiffness of the original system.

Figure 1 shows a comparison of the exact and numerical solutions for the profile of the longitudinal component of velocity in the section at x=6.6. The velocity profile converges to the exact solution.

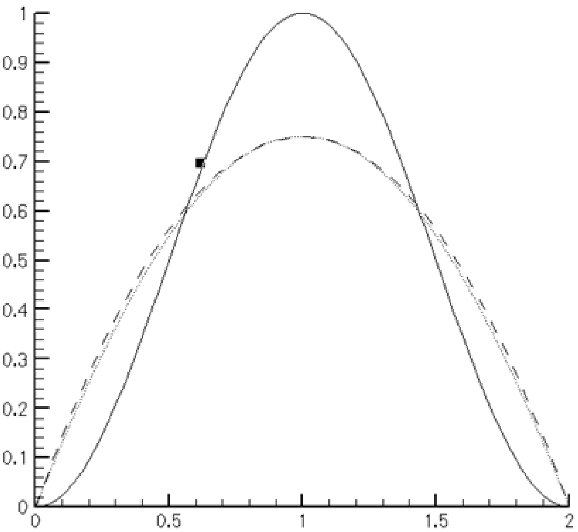

Figure 1 – Comparison of numerical results with exact solution for Poiseuille flow calculation: velocity profiles " " - initial velocity; "- - - - - -" - exact solution; "........" - numerical solution

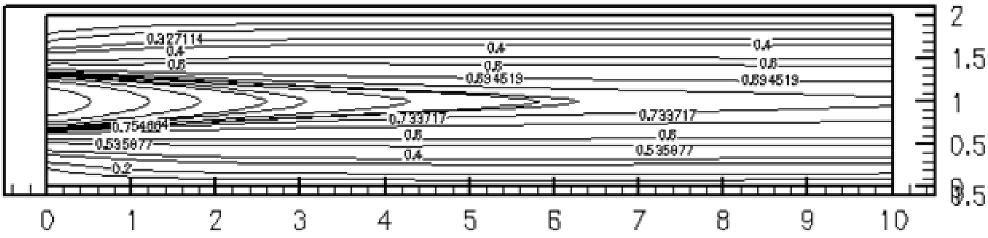

Figure 2 – Velocity component u in section xz

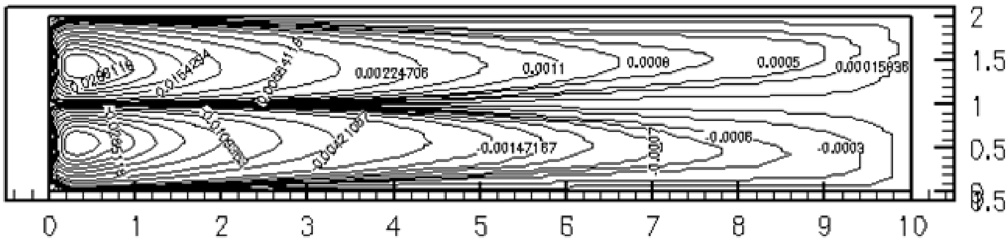

Figure 3 – Velocity component w in section xz

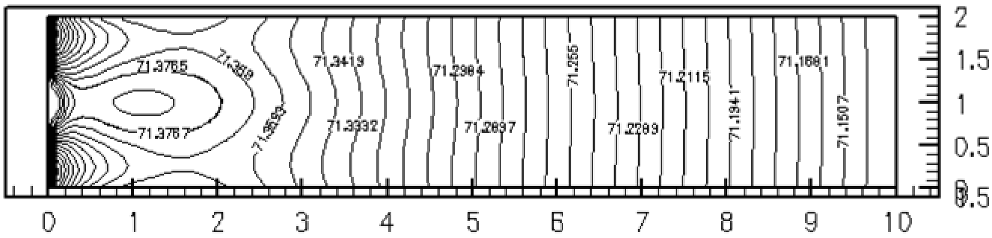

Figure 4 – Pressure in section xz

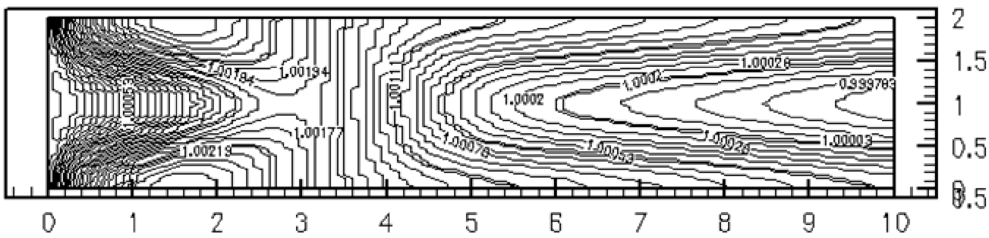

Figure 5 – temperature in section xz

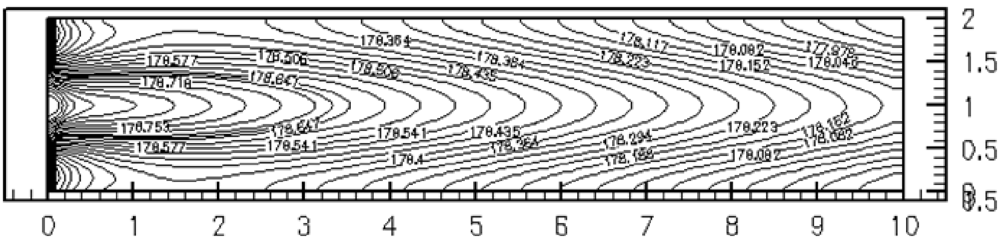

Figure 6 – total energy in section xz

The velocity profiles (Figures 2-3) quickly approache the exact solution as shown in Figure 1 [19]. Accurate results are ensured by the boundary conditions. The pressure plot (Figure 4) shows a constant pressure gradient (Figure 4) in most channels, which is $0.998(\partial p^{exact}/\partial x_1)$. The boundary layer behaviour is not evident in the longitudinal velocity profiles (Figure 2) or in the temperature regime (Figure 5).

It can be concluded that, the convergence rate of the solution is high enough, strong pressure and temperature gradients are formed near the outflow, and a boundary layer appears in the temperature contours.

Conclusion

Based on the analysis of the existing literature, the state of the problem of numerical solution of the full Navier-Stokes equations for compressible turbulent viscous gas at small Mach numbers is studied. It is known that the main problem in solving finite-difference equations is numerical instability leading to the problem of convergence of the iterative process. The main ways of eliminating the arising problems are studied. It is obtained that, to date, there are two main approaches in solving the initial equations. The Density-based method, where the full Navier-Stokes equations are used and pressure, temperature and entropy are solved directly using the basic parameters. It was found that within the Density-based method, the most common way to eliminate computational difficulties at small Mach numbers is to apply preconditioning of the initial equations, which allows the difference equations to be modified in such a way that the eigenvalues of the Jacobi matrix (wave propagation velocity) of the modified system of equations are of the same order. It is revealed that the main advantage of the preconditioning method is that it allows to solve flow problems with both small and moderate Mach numbers in a unified way.

The mathematical formulation of the problem of turbulent air flow with transverse jet induction in a substantially subsonic regime with the use of the preconditioning method has been formulated. The main advantage of using the procedure of preconditioning when solving the original Favre-averaged Navier-Stokes equations is the possibility of modifying the difference equations in such a way that the eigenvalues of the Jacobian (wave propagation velocity) of the modified system of equations have the same order. A numerical method for solving the problem is developed, in which the Turkel preconditioning matrix for primitive variables is used. The transition to conservative variables is performed using transition matrices, and the right and left eigenvectors are calculated for the system of three-dimensional preconditioning matrices.

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ЖЫЛДАМДЫҒЫ ДЫБЫС ЖЫЛДАМДЫҒАНАН ТӨМЕН АҒЫНДАР ҮШІН ШАРТТЫ ТҮРЛЕНДІРУ ӘДІСІ

Аңдатпа

Ұсынылып отырған жұмыс қабырғада пайда болған симметриялық перпендикуляр ағындарға назар аудара отырып, арнадағы үш өлшемді жылдамдығы дыбыс жылдамдығынан төмен турбулентті ағынДЫ МҰҚИЯТ сандық модельдеуге арналған. Турбуленттілік моделімен тұйықталған Фавр-орташаланған Навье-Стокс теңдеулері ENO схемасына негізделген алгоритм арқылы шешіледі. Итерациялық үрдістің конвергенциясын жылдамдату үшін шартты түрлендіру әдісі қолданылады және қарабайыр айнымалылар векторына көшу жүзеге асырылады. Зерттеу нәтижелері дыбыстық турбулентті ағындарды жақсырақ түсіну үшін маңызды және әртүрлі салаларда, соның ішінде инженерлік және ғылыми зерттеулерде қолданылуы мүмкін. Жұмыстың өзектілігі жоғары ретті дәлдік схемаларын пайдалана отырып, үш өлшемді Навье-Стокс теңдеулерін шешуге мүмкіндік беретін тиімді сандық алгоритмдерді әзірлеумен, сондай-ақ дыбыстан жоғары көп компонентті ағынды талдау үшін сенімді турбуленттілік модельдерін қолданумен ерекшеленді. Жұмыстың ғылыми жаңалығы итерациялық процестің конвергенциясын жеделдету үшін шартты түрлендіру әдісін сәтті қолдануында жатыр.

Тірек сөздер: сандық модельдеу, жылдамдығы дыбыс жылдамдығынан төмен ағындар, идеал газ, шекаралық қабат, Навье-Стокс теңдеулері.

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МЕТОД ПРЕДОБУСЛАВЛИВАНИЯ ДЛЯ СУЩЕСТВЕННО ДОЗВУКОВЫХ ПОТОКОВ

Аннотация

Предлагаемая работа представляет собой детальное численное моделирование трехмерного дозвукового турбулентного течения в канале с основным упором на симметричные перпендикулярные струи, возникающие у стенок. Решение осредненных по Фавру уравнений Навье-Стокса, замкнутых моделью турбулентности, осуществляется с помощью алгоритма, основанного на схеме ENO. Для ускорения сходимости итерационного процесса используется метод предобуславливания и осуществляется переход к вектору примитивных переменных. Результаты исследования важны для лучшего понимания дозвуковых турбулентных потоков и могут найти применение в различных областях, включая инженерные и научные исследования. Актуальность работы подчеркивается разработкой эффективных численных алгоритмов, позволяющих решать дозвуковые трехмерные уравнения Навье-Стокса с использованием схем высокого порядка точности, а также с применением робастных моделей турбулентности для анализа сверхзвукового многокомпонентного течения. Научная новизна работы заключается в успешном использовании метода предобусловливания для ускорения сходимости итерационного процесса.

Ключевые слова: численное моделирование, дозвуковое течение, идеальный газ, пограничный слой, уравнения Навье-Стокса.