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FRACTAL GEOMETRY AND LEVEL SETS INCONTINUED FRACTIONS

Abstract

Continued fractions offer a unique representation of real numbers as a sequence of natural numbers. Good's seminal work on continued fractions laid further research into fractal geometry and exceptional sets. This paper extends Good's findings by focusing on level sets constructed by restricting the partial quotients with lower bounds. Using elementary approaches, we establish new bounds on their Hausdorff dimension, providing theoretical insights and practical estimation methods. Additionally, we offer alternative proofs and corollaries that deepen our understanding of the relationship between continued fractions and fractal geometry. Continued fractions provide a distinctive means of expressing real numbers as a sequence of natural numbers, offering insights into the underlying structure of these numbers. Building upon Good's foundational research in continued fractions, this paper delves into the domain of fractal geometry and exceptional sets, exploring the interesting connections between these mathematical constructs. Our focus lies on investigating the Hausdorff dimension of level sets formed by constraining the partial quotients with lower bounds. Employing elementary methodologies, we present fresh theoretical bounds on Hausdorff dimension of these level sets, enriching our understanding of their geometric properties. Through combining theoretical advancements and practical techniques, this research contributes to mathematics, providing both deep theoretical insights and practical applications in understanding continued fractions and their geometric properties.

Key words: continued fractions, number theory, Hausdorff dimension, fractals, numerical approximation, Newton–Raphson method, Taylor series.

Introduction

Continued fractions are a unique way of representing a real number x in $[0,1)$ as a sequence $(a_n(x))$ of natural numbers:

$$x = \cfrac{1}{a_1(x)+\cfrac{1}{a_2(x)+\cfrac{1}{a_3(x)+\cdots}}} \text{ or } [a_1, a_2, a_3, \cdots]$$

They can be obtained through an iterative process of representing a number as the sum of its integer part and the reciprocal of the fractional part. This process can be finite or infinite, and it provides an alternative representation of real numbers, which has connections to various areas of mathematics such as number theory, hyperbolic geometry, and Diophantine approximation. Continued fractions have applications in various fields, including the analysis of rational approximations, the study of quadratic irrationals, and the computation of irrational numbers like π. The study of continued fractions also has connections to fractal geometry and dimension theory. Fractals are complex geometric shapes that exhibit self-similarity at different scales and have a non-integer dimension. The connection between continued fractions and fractal geometry lies in self-similar properties of certain continued fractions, which can lead to the exploration of fractal sets with specific dimensions, such as the Box dimension and Hausdorff dimension [1]. Therefore, the study of continued fractions and fractal dimension provides a rich and interconnected area of mathematical research, with implications for diverse fields within mathematics and its applications.

In his work, Good [2] studied the Hausdorff dimension of various exceptional sets arising in continued fractions and in particular proved the following:

For $\alpha \geq 20,$ the Hausdorff dimension of exceptional sets

$$\frac{1}{2}+\frac{1}{2log(\alpha+2)} < dim_H \{x \in [0,1] \mid a_n(x) \geq \alpha, \forall n\} < \frac{1}{2}+\frac{log\,log(\alpha-1)}{2log(\alpha-1)}.$$

Here we denote log as the natural logarithm.

Let us define the set F of divergent partial quotients, namely

$$F = \{x \in [0,1] \mid a_n(x) \to \infty \; as \; n \to \infty\}.$$

He also showed that

$$dim_H F = \frac{1}{2}.$$

Numerous studies have been undertaken to generalize and build upon the findings since Good initially published his work. In two consecutive works, Hirst investigated the dimension of sets with partial quotients satisfying functions $a_n \geq f(n)$ tending to infinity [3-4]. Specifically, he demonstrated that the dimension of the set of real numbers with partial quotients satisfying $a_n \geq n^b$ is 1/(2b). Subsequently, Cusick demonstrated that for given values of a and b greater than 1, the set with $a_n \geq a^{b^n}$ for all n has a zero Hausdorff dimension [5]. Later Fan et al [6] obtained a general framework for calculating the dimension of the sets satisfying $s_n \leq a_n(x) \leq N s_n$ with any increasing sequence s_n. Another avenue of generalization involved considering $limsup$ sets rather than all statements. In this regard, Moorthy [7] established that the set with $a_n \geq a^{b^n}$ for infinitely many n has a dimension of at most 2/(1+b). He hypothesized that the exact dimension should be 1/(1+b), which was later confirmed by Luczak in 1997 [8]. Additionally, refer to [9] for an alternative proof of the lower bound. For further results in this area, please consult [10–12].

In recent years, there has been a notable increase in research focusing on the dimensions of divergent partial quotients. Particularly, the study of weakly divergent partial quotients was undertaken in [13]. Additionally, investigations into increasing partial quotients with joint constraints were conducted in both [14] and [15]. In [16–17], various formulas were derived pertaining to the dimension of sets where the growth rate of partial quotients a_n is comparable to $e^{\phi(n)}$, where $\phi(n)$ satisfies various conditions. Furthermore, in [18], the authors explored exceptional sets wherein the limit inferior and limit superior of $log\ a_n/log\ n$ converge to a and b respectively. They demonstrated that for positive values of a, the dimension is one half, and it is 1 if a equals zero. Takashi [19] delved into the study of slowly divergent partial quotients.

From the literature, it is evident that while numerous works have concentrated on sets with divergent partial quotients, there appears to be a gap in research regarding the dimension of level sets with lower bounds since Good's work. In this work, we plan to focus on this direction of research.

For any positive α we define the following level set

$$F_\alpha = \{x \in [0,1) \mid a_n(x) \geq \alpha, \forall n \geq 1\}.$$

In this work we consider an elementary approach in estimating the dimension of the level sets F_α and improvement of the Good's above-mentioned results. Our main result is the following.

Theorem 1. Let $\alpha \geq 2$ be an integer. Let $\epsilon(\alpha) \in (0,1)$ be a real number such that

$$\epsilon(\alpha)\,(\alpha-1)^{\epsilon(\alpha)} > 1 \tag{1}$$

Then, the Hausdorff dimension of F_α satisfies

$$dim_H F_\alpha < \frac{1}{2} + \frac{1}{2}\epsilon(\alpha).$$

For $\epsilon(\alpha) = \frac{loglog(\alpha-1)}{log(\alpha-1)}$ with $\alpha \geq 20$ it is easy to see that the condition (1) is satisfied. Hence, this theorem can be thought as an improvement of Good's result. We note that we do not have any restrictions to α other than it being at least 2. Clearly, when $\alpha = 1$, the theorem is false, as in this case $F_1 = [0,1)$ which has full Hausdorff dimension 1. In fact, it has Lebesgue measure 1.

In the next, we consider various estimates for $\epsilon(\alpha)$. Using the Newton–Raphson method we obtain the following corollary.

Corollary 2. Let $\alpha \geq 17$ be an integer. Then, the Hausdorff dimension of F_α satisfies

$$dim_H F_\alpha < \frac{1}{2} + \frac{1}{2}\frac{(log\ log\ (\alpha-1))^2 + 1}{log\ (\alpha-1)\ (log\ log\ (\alpha-1) + 1)}.$$

Another approach to estimate $\epsilon(\alpha)$ is to use Taylor's approximation. In this case, we have the following corollaries:

Corollary 3. Let $\alpha \geq 2$ be an integer. Then, the Hausdorff dimension of F_α satisfies

$$dim_H F_\alpha < \frac{1}{2} + \frac{-1 + \sqrt{1 + 4\,log\ (\alpha-1)}}{4\,log\ (\alpha-1)}.$$

Corollary 4. Let $\alpha \geq 2$ be an integer. Then, the Hausdorff dimension of F_α satisfies

$$dim_H F_\alpha < \frac{1}{2} + \frac{1}{2}\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \frac{1}{2}\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{1}{2}\sqrt{\frac{2p}{3}},$$

where $p = \frac{2}{3log^2(\alpha-1)}$ and $q = -\frac{20+54\,log(\alpha-1)}{27log^3(\alpha-1)}$.

As a by-product, we obtain another proof of Good's result that the dimension of the set of divergent partial quotients is one-half.

Main Provisions

Define the Hausdorff dimension:

Definition 1. Recall that a δ-cover of a set F is a countable (or finite) collection of sets $\{U_i\}$ with diameters $0 < |U_i| \le \delta$ that cover F. Suppose that $F \subset R^n$ and $s \ge 0$. For each $\delta > 0$, we define

$H_\delta{}^s(F) = inf\ \{\sum_{i=1}^{\infty} |U_i|^s :\ \{U_i\}\ is\ a\ \delta - cover\ of\ F\}.$

We call $H^s(F) = \lim_{\delta \to 0} H_\delta{}^s(F)$ the s-dimensional Hausdorff measure of F.

Definition 2. Let $H^s(F)$ be a Hausdorff measure, then Hausdorff dimension of a set F is

$dim_H\, F = inf\ \{s \ge 0 :\ H^s(F) = 0\} = sup\ \{s > 0 :\ H^s(F) = \infty\}.$

For any continued fraction $[a_1, a_2, a_3, \cdots]$ we define the convergents p_n/q_n by

$$\frac{p_n}{q_n} = [a_1, a_2, a_3, \cdots, a_n],$$

where p_n, q_n are in lowest terms, so that they are coprime positive integers. In fractal geometry to compute the dimension of the given set one often needs to find a minimal number of intervals needed to cover the given set and the lengths of these intervals. The following lemma gives us these building blocks.

Lemma 1. For any natural numbers $n, a_1, a_2, \cdots, a_n$ we let $I_n(a_1, a_2, \cdots, a_n)$ the following set $I_n(a_1, a_2, \cdots, a_n) = \{x \in [0,1] \mid a_1(x) = a_1, a_2(x) = a_2, \cdots, a_n(x) = a_n\}$.

$I_n(a_1, a_2, \cdots, a_n) = (\frac{p_n}{q_n}, \frac{p_n + p_{n-1}}{q_n + q_{n-1}})\ or\ (\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \frac{p_n}{q_n})$

The lemma is well-known and for the proof we refer to [20].

Lemma 2. For any natural numbers $n, a_1, a_2, \cdots, a_n$, the intervals have the following length estimate

$$|I_n(a_1, a_2, \cdots, a_n)| \le (a_1 a_2 \cdots a_n)^{-2}.$$

$$|I_n(a_1, a_2, \cdots, a_n)| = |\frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{p_n}{q_n}| = \frac{|p_{n-1}q_n - p_n q_{n-1}|}{q_n(q_n + q_{n-1})}.$$

Since $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$, see e.g. [21], we have

$$|I_n(a_1, a_2, \cdots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})}$$

We now obtain the growth rate of q_n using the fact that the convergents can be constructed using difference equations [21]:

$$q_n = a_n q_{n-1} + q_{n-2} \ge a_n q_{n-1} = a_n(a_{n-1}q_{n-2} + q_{n-3}) \ge a_n a_{n-1} q_{n-2} \ge \ldots \ge a_n a_{n-1} \ldots a_1.$$

As a result, we have the following length estimate

$$|I_n(a_1, a_2, \cdots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})} \le \frac{1}{(q_n)^2} \le \frac{1}{(a_1 a_2 \ldots a_n)^2}.$$

Materials and methods

In this section we prove our results stated in the introduction.

Proof of Theorem 1. From the definition of Hausdorff dimension we deduce that if there exists s such that $H^s(F_\alpha) = 0$, then $dim_H\ (F_\alpha)\ \leq s.$ Since $H_\delta{}^s(F_\alpha)$ is an infimum of sums, we see that for any family of δ-covers $\{U_i{}^\delta\}$ of F_α it holds

$$H^s(F_\alpha) = \lim_{\delta\to 0} H_\delta{}^s(F_\alpha) \leq \lim_{\delta} \textstyle\sum_i |U_i{}^\delta|^s.$$

$$F_\alpha \subset \bigcup_{a_i\geq\alpha} I_n(a_1, a_2, \cdots, a_n),$$

where the union is taken over all possible integer tuples $a_1, a_2, \cdots, a_n\ \geq \alpha.$ This is a countable set and $\{I_n(a_1, a_2, \cdots, a_n)\}$ provides a cover for F_α. Hence for a suitable δ it follows that

$$H_\delta{}^s(F_\alpha) \leq \textstyle\sum_{a_1,\dots,a_n\geq\alpha} |I_n(a_1, a_2, \cdots, a_n)|^s \leq \sum_{a_1,\dots,a_n\geq\alpha} (a_1 a_2 \dots a_n)^{-2s} \leq (\sum_{k=\alpha}^{\infty} k^{-2s})^n$$

For the sum to converge, we need $s > 1/2.$ We may recognize the sum as the lower Riemann sum for an integral:

$$(\textstyle\sum_{k=\alpha}^{\infty} k^{-2s})^n \leq (\int_{\alpha-1}^{\infty} x^{-2s} dx)^n = (\frac{1}{(2s-1)(\alpha-1)^{2s-1}})^n,$$

We note that as n increases, the sizes of the intervals $I_n(a_1, a_2, \cdots, a_n)$ decreases to zero, hence δ decreases to zero. Thus, to obtain that $H^s(F_\alpha) = 0$ we need to take s that depends on α such that $(2s-1)(\alpha-1)^{2s-1} > 1.$ For such s we then have $dim_H\ (F_\alpha)\ \leq s.$ Making a change of variable $\epsilon(\alpha) = 2s - 1$ we see that once $\epsilon(\alpha)(\alpha-1)^{\epsilon(\alpha)} > 1$ we have

$$dim_H\ (F_\alpha)\ \leq s = \frac{1}{2} + \frac{1}{2}\epsilon(\alpha).$$

We do not turn in proving the corollaries.

Proof of Corollary 2. To prove the statement, we utilize the Newton-Raphson method. The Newton-Raphson method, a powerful tool for solving nonlinear equations, has been the subject of various studies [22]. Consider the function $f(x) = x(\alpha-1)^x - 1.$ In view of Theorem 1 we need to approximate the root of the function $f(x)$ from above. The graph of the function for several values of α is depicted in Figure 1 (p. 30).

For $x = \frac{\log\log(\alpha-1)}{\log(\alpha-1)} = \log_{\alpha-1}\log(\alpha-1)$ we see that

$$f(x) = \frac{\log\log(\alpha-1)}{\log(\alpha-1)}(\alpha-1)^{\frac{\log\log(\alpha-1)}{\log(\alpha-1)}} - 1 = \log\log\ (\alpha-1) - 1.$$

Since $\log\log\ 16 \approx 1.01978,$ it follows that for $\alpha\ \geq 17$ we have

$\log\log\ (\alpha-1) - 1\ >\ 0$. In particular, if we initiate the Newton–Raphson method with the initial value $x_0 = \frac{\log\log(\alpha-1)}{\log(\alpha-1)}$, then in any iteration we remain on the right-hand side of the x-intercept due to convexity. We note that the derivative of the function is

$$f'(x) = (1 + x\log\ (\alpha-1))(\alpha-1)^x.$$

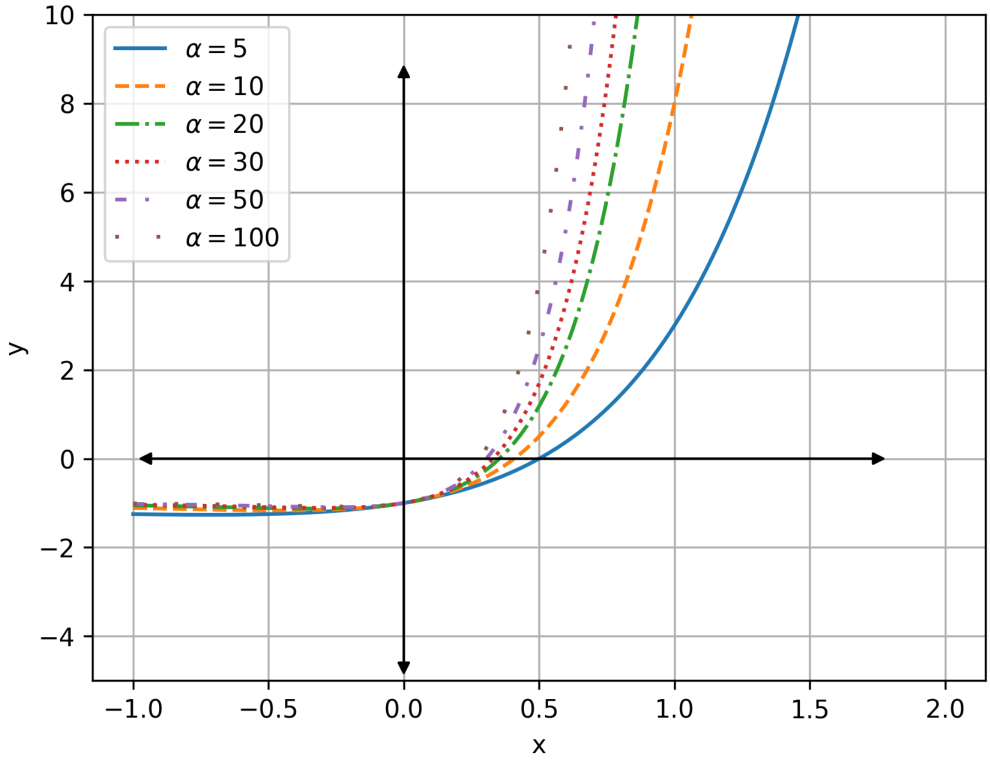

Figure 1. The graph of $f(x)$ for various values of α.

In one iteration of the Newton–Raphson method we get

$$\epsilon(\alpha) := x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0(\alpha-1)^{x_0}-1}{(\alpha-1)^{x_0}(x_0 log(\alpha-1)+1)}.$$

$$\epsilon(\alpha) = \frac{(log\ log\ (\alpha-1))^2+1}{log(\alpha-1)(log\ log\ (\alpha-1)+1)}.$$

Proof of Corollary 3. Our goal is to estimate $\epsilon(\alpha)$ from above such that $\epsilon(\alpha)(\alpha-1)^{\epsilon(\alpha)} > 1$. To this end, we use Taylor approximation. We know that e^x has Taylor expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Writing $(\alpha-1)^x = e^{x\,log(\alpha-1)}$ we see that the Taylor expansion for $x(\alpha-1)^x$ is given by

$$x(\alpha-1)^x = x\sum_{n=0}^{\infty} \frac{(x\ log\ (\alpha-1))^n}{n!} = \sum_{n=0}^{\infty} \frac{log\ (\alpha-1))^n}{n!} x^{n+1}.$$

Considering 2nd order approximation we see that

$$x(\alpha-1)^x = x + log\ (\alpha-1)x^2 + O(x^3).$$

From this it follows that if $x + log\ (\alpha-1)x^2 = 1$, then $x(\alpha-1)^x = 1 + O(x^3) > 1$ which gives an upper estimate. Solving the quadratic equation

$$log\ (\alpha-1)x^2 + x - 1 = 0$$

we get $\epsilon(\alpha) = x = \frac{-1+\sqrt{1+4\ log\ (\alpha-1)}}{2\ log\ (\alpha-1)}$. Now, applying Theorem 1 yields the desired estimate.

Proof of Corollary 4. Arguing as in the proof of Corollary 4, but this time considering 3rd order Taylor approximation we see that the positive solution to

$$x + log\,(\alpha-1)x^2 + \frac{log^2\,(\alpha-1)}{2}x^3 = 1$$

within the interval $(0,1)$ gives a good estimate for $\epsilon(\alpha)$. Let us analyze the function

$$f(x) = \frac{log^2(\alpha-1)}{2}x^3 + log\,(\alpha-1)x^2 + x - 1$$

Taking derivative, we get $f'(x) = \frac{3\,log^2(\alpha-1)}{2}x^2 + 2\,log\,(\alpha-1)x + 1$. From Arithmetic mean – Geometric mean inequality we get

$$\frac{3\,log^2\,(\alpha-1)}{2}x^2 + 1 \geq 2\sqrt{\frac{3\,log^2\,(\alpha-1)}{2}x^2} > 2\,log\,(\alpha-1)|x|.$$

It follows that $f'(x) > 0$ on the real line so that the function is strictly increasing. Since any cubic function has at least one root, we deduce that the function $f(x)$ has exactly one real root. Also, since $f(0) < 0 < f(1)$ we conclude that for any positive integer $\alpha \geq 2$ the unique real root must be on the interval $(0,1)$. It can be found using Cardano's formulas

$$\epsilon(\alpha) = x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt{\frac{2p}{3}},$$

where $p = \frac{2}{3log^2(\alpha-1)}$ and $q = -\frac{20+54\,log(\alpha-1)}{27log^3(\alpha-1)}$.

Results and Discussion

In this section we compare our results to that of Good. To this end, we provide various graphical illustrations. In Figure 2, we compared the Hausdorff dimension estimates for F_α for various α ranging from 2 to 500 using different methods, namely 1st order Newton-Raphson, 2nd order Taylor expansion, and 3rd order Taylor expansion, against Good's upper bounds. Second order Taylor, as depicted in the graph, shows that it consistently remains above all other curves, indicating an overestimation of the Hausdorff dimension. Our remaining two estimation methods, 1st order Newton-Raphson and 3rd order Taylor expansion, on the other hand, stays lower than Good, suggesting an improved result. The best upper estimate for the Hausdorff dimension among these methods turns out to be 1st order Newton-Raphson with initial value $\frac{log\,log\,(\alpha-1)}{log(\alpha)}$. We note that clearly one can improve both Taylor and Newton-Raphson, however the dimension formulas get very complicated. These findings provide valuable insights into the performance and accuracy of different methods for estimating the Hausdorff dimension.

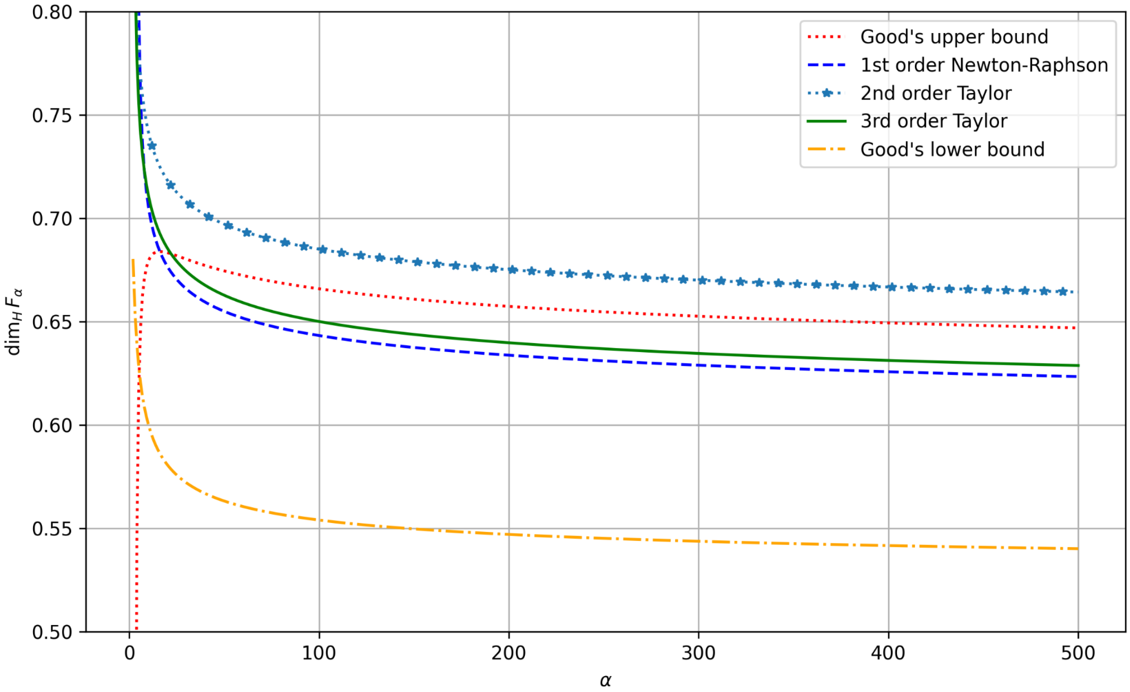

Figure 2. Dimension upper estimates of F_α for varying α

Conclusion

The exploration of continued fractions and their connection to fractal geometry has yielded significant advancements in mathematical theory and practical applications. Through this paper, we have extended Good's seminal work on continued fractions and exceptional sets, focusing particularly on level sets constructed by imposing lower bounds on partial quotients.

Our main result, Theorem 1 and its corollaries, provide new bounds on the Hausdorff dimension of these level sets, improving upon Good's earlier findings. In our corollaries, we offer practical estimation methods for the Hausdorff dimension of these sets. Utilizing approaches such as the Newton-Raphson method and Taylor series approximation, we provide insights into the behavior of these level sets under different constraints and parameters. By establishing conditions under which these level sets exhibit specific dimensional properties, we deepen our understanding of the intricate relationship between continued fractions and fractal geometry. Additionally, our graphical analysis comparing our estimations with Good's upper bounds highlights the effectiveness of our methods in providing accurate dimension estimates.

Overall, our obtained results contribute to the ongoing research in this field, shedding light on the complex structures inherent in continued fractions and their implications for fractal geometry. Further investigations into the properties of level sets with lower bounds present avenues for future exploration and refinement of dimension estimation techniques. Besides, similar numerical approaches can be utilized to better approximate the Hausdorff dimension of the level sets from below.

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ҮЗДІКСІЗ БӨЛШЕКТЕРДЕГІ ФРАКТАЛДЫ ГЕОМЕТРИЯ ЖӘНЕ ДЕҢГЕЙЛІ ЖИЫНДАР

Аңдатпа

Үздіксіз бөлшектер натурал сандар тізбегі ретінде нақты сандардың бірегей бейнесін түсінуге көмектеседі. Гудтың үздіксіз бөлшектер туралы негізгі жұмысы фракталды геометрия мен ерекше жиындардың әрі қарай зерттелуіне түрткі болды. Бұл мақалада төменгі шектермен шектеу арқылы алынған деңгей жиындарына баса назар аудару арқылы Гудтың нәтижелері одан әрі жетілдірілген. Қарапайым тәсілдер арқылы біз теориялық білім мен тәжірибелік бағалау әдістерін қолдана отырып, олардың Хаусдорф өлшемі үшін жаңа шекараларын орнатамыз. Сонымен қатар біз үздіксіз бөлшектер мен фракталдық геометрия арасындағы байланыс туралы түсінігімізді тереңдететін балама дәлелдер мен оның салдарын ұсынамыз. Үздіксіз бөлшектер нақты сандарды натурал сандар тізбегі ретінде өрнектеудің ерекше тәсілін ұсынады, бұл өз кезегінде осы сандардың негізгі құрылымын жақсырақ түсінуге көмектеседі. Гудтың үздіксіз бөлшектердегі іргелі зерттеулеріне сүйеніп, осы математикалық конструкциялар арасындағы қызықты байланыстарды зерттей отырып, бұл мақала фракталдық геометрия мен ерекше жиындар салаларына терең көз жүгіртеді. Біздің басты назарымыз үздіксіз бөлшектерді төменгі шектермен шектеу арқылы құрылған деңгей жиындарының Хаусдорф өлшемін зерттеу. Элементар әдістемелерді қолданып, біз олардың геометриялық қасиеттері туралы түсінігімізді кеңейте отырып, деңгей жиындарының Хаусдорф өлшеміне жаңа теориялық шекараларды ұсынамыз. Теориялық жетістіктер мен практикалық әдістерді біріктіре отырып, бұл зерттеу терең теориялық білім мен үздіксіз бөлшектер және олардың геометриялық қасиеттерін түсіну үшін практикалық қолданбаларды қамтамасыз ету арқылы математикаға үлес қосады.

Тірек сөздер: үздіксіз бөлшектер, сандар теориясы, Хаусдорф өлшемі, фракталдар, сандық жуықтау, Ньютон-Рафсон әдісі, Тейлор қатары.

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ГЕОМЕТРИЯ ФРАКТАЛОВ И МНОЖЕСТВА УРОВНЕЙ В ЦЕПНЫХ ДРОБЯХ

Аннотация

Цепные дроби предлагают уникальное представление действительных чисел в виде последовательности натуральных чисел. Основополагающая работа Гуда о цепных дробях положила начало дальнейшим исследованиям фрактальной геометрии и исключительных множеств. Эта статья расширяет выводы Гуда, сосредоточив внимание на множествах уровня, построенных путем ограничения частичных дробей нижними границами. Используя элементарные подходы, мы устанавливаем новые границы их хаусдорфовой размерности, предоставляя теоретические знания и практические методы оценки. Кроме того, мы предлагаем альтернативные доказательства и следствия, которые углубляют наше понимание взаимосвязи между цепными дробями и фрактальной геометрией. Цепные дроби представляют особый способ выражения действительных чисел в виде последовательности натуральных чисел, что позволяет лучше понять основную структуру этих чисел. Основываясь на фундаментальных исследованиях Гуда в области цепных дробей, эта статья углубляется в область фрактальной геометрии и исключительных множеств, исследуя интересные связи между этими математическими конструкциями. Наше внимание сосредоточено на исследовании хаусдорфовой размерности множеств уровня, образованных путем ограничения частичных дробей нижними границами. Используя элементарные методологии, мы представляем новые теоретические границы хаусдорфовой размерности этих множеств уровней, обогащая наше понимание их геометрических свойств. Сочетая теоретические достижения и практические методы, это исследование вносит вклад в математику, предоставляя как глубокие теоретические знания, так и практические приложения для понимания цепных дробей и их геометрических свойств.

Ключевые слова: цепные дроби, теория чисел, размерность Хаусдорфа, фракталы, численное приближение, метод Ньютона-Рафсона, ряд Тейлора.