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## INITIAL-BOUNDARY VALUE PROBLEMS TO THE TIME-NONLOCAL DIFFUSION EQUATION

**Abstract**

This article investigates a fractional diffusion equation involving Caputo fractional derivative and Riemann-Liouville fractional integral. The equation is supplemented by initial and boundary conditions in the domain defined by the interval by space $0 < x < 1$ and interval by time $0 < t < T$. The fractional operators are defined rigorously, utilizing the Caputo fractional derivative of order $\beta$ and the Riemann-Liouville fractional integral of order $\alpha$, where $0 < \alpha < \beta \leq 1$. The main results include the presentation of well-known properties associated with fractional operators and the establishment of the unique solution to the given problem. The key findings are summarized through a theorem that provides the explicit form of the solution. The solution is expressed as a series involving the two-parameter Mittag-Leffler function and orthonormal eigenfunctions of the Sturm-Liouville operator. The uniqueness of the solution is proven, ensuring that the problem has a single, well-defined solution under specific conditions on the initial function. Furthermore, the article introduces and proves estimates related to the Mittag-Leffler function, providing bounds crucial for the convergence analysis. The convergence of the series is investigated, and conditions for the solution to belong to a specific function space are established. The uniqueness of the solution is demonstrated, emphasizing its singularity within the given problem. Finally, the continuity of the solution in the specified domain is confirmed through the uniform convergence of the series.

**Key words:** fractional derivative, integral equation, the method of separation variables, time-nonlocal diffusion equation.

## Introduction

Over the course of millennia, fractional partial differential equations (FPDEs) have evolved into essential tools for representing complex systems and anomalous phenomena [1]-[3]. A comprehensive exploration of the applications of these equations across disciplines such as chemistry, technology, and physics is presented in the book [4]. Notably, the book discusses the utilization of fractional derivatives to modify the classical diffusion equation, resulting in the equation of fractional diffusion in time. Additionally, [5] investigates initial-boundary value problems for the diffusion equation with variable coefficients, considering both Dirichlet and Neumann conditions.

In [6], Luchko extends the maximum principle to the generalized diffusion equation involving a fractional time derivative. This extension is applied to establish uniqueness and existence results for the initial-boundary value problem associated with the fractional diffusion equation.

The work in [7] focuses on exploring the generalized solution for the initial-boundary value problem of the diffusion equation with fractional time. Fractional calculus has emerged as a powerful tool for modeling and analyzing complex phenomena in various scientific disciplines. In this article, we delve into the realm of fractional partial differential equations, specifically exploring a novel equation involving Caputo fractional derivative and Riemann-Liouville fractional integral.

The equation, defined over the domain $\Omega = \{(x,t): 0 < x < 1, 0 < t < T\}$, is accompanied by carefully crafted initial and boundary conditions. Motivated by the intricate nature of fractional operators, we introduce the Caputo fractional derivative of order $\beta$ and the Riemann-Liouville fractional integral of order $\alpha$. These operators play a pivotal role in formulating and solving the fractional partial differential equation under consideration. To establish the groundwork, we present fundamental properties associated with these fractional operators, drawing upon existing literature [9, 10, 11, 12, 13, 14].

In summary, this article navigates through the complexities of fractional calculus, unraveling the unique features and behaviors of the presented partial differential equation. The insights gained here pave the way for a deeper comprehension of fractional operators and their applications in mathematical modeling.

In this article we consider the following equation

$$\left(D_{0+}^{\beta}u\right)(x,t) - \frac{\partial^2}{\partial x^2}(I_{0+}^{\alpha}u)(x,t) = 0, \text{ in } \Omega = \{(x,t): 0 < x < 1, 0 < t < T\} \quad (1.1)$$

with initial and boundary conditions

$$u(x,0) = \varphi(x) \text{ on } x \in [0,1], \quad (1.2)$$

$$u(0,t) = u(1,t) = 0, \ 0 \le t \le T, \quad (1.3)$$

where $0 < \alpha < \beta \le 1$ and the function $\varphi$ is continuous. The operator $D_{0+}^{\beta}$ stands for the Caputo fractional derivative of order $\beta \in (0,1)$ is defined by

$$\left(D_{0+}^{\beta}u\right)(x,t) = I_{0+}^{1-\beta}\left[\frac{\partial}{\partial t}u(x,t)\right] = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\frac{\partial}{\partial s}u(x,s)ds$$

and the operator $I_{0+}^{\alpha}$ is the Riemann-Liouville fractional integral of order $\alpha > 0$, defined as

$$(I_{0+}^{\alpha}u)(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(x,s)ds, t \in (0,T].$$

The case when, instead of the operator $I^{\alpha}_{0+}$, the time-degenerate diffusive coefficient $t^{\beta}$ with $\beta > -\alpha$ is used, studied for the one-dimensional linear time-fractional diffusion equation in [9]

$$\left(D^{\beta}_{0+}u\right)(x,t) - t^{\beta}u_{xx}(x,t) = 0 \text{ in } (x,t) \in \mathbb{R} \times (0,\infty).$$

The authors have found an explicit solution by using the Kilbas-Saigo function. Moreover, the convergence, the existence and uniqueness of the solution of the problem are confirmed.

Solving such problems may involve using techniques like Laplace transforms, Fourier transforms, or other integral transforms to handle the nonlocal term. The well-known traits associated with fractional operators are presented below [15, 16, 17, 18].

**Main provisions. Material and methods**

The main purpose of this article is to present the key conclusions related to the classical solution. A central theorem provides an explicit expression for the solution, revealing a series representation involving the two-parameter Mittag-Leffler function. The uniqueness of the solution is rigorously proven, contingent upon specific conditions governing the initial function.

Furthermore, we introduce and prove estimates for the Mittag-Leffler function, crucial for understanding its behavior and ensuring convergence. The convergence of the series solution is scrutinized, and conditions for the solution to reside in a particular function space are derived. Emphasis is placed on the singularity of the solution within the given problem.

In the subsequent sections, we delve into the proof of continuity for the solution in the specified domain, demonstrating its uniform convergence. These results contribute significantly to the broader understanding of fractional partial differential equations and shed light on the intricacies associated with the involved operators [19,20].

**Lemma 1.1.** [3, P. 95] If $0 < \beta < 1$ for $T \in AC[0,T]$ or $T \in C'(0,T)$, then

$$I^{\beta}_{0+}\left[\left(D^{\beta}_{0+}T\right)(t)\right] = T(t) - T(0)$$

holds true.

**Lemma 1.2.** [3, P. 101] Let $T \in C[0,T]$. If $\alpha + \beta \leq 1$, then

$$I^{\beta}_{0+}[(I^{\alpha}_{0+}T)(t)] = \left(I^{\alpha+\beta}_{0+}T\right)(t).$$

Next, we have an estimate of the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(-z)$.

**Lemma 1.3.** [8, P. 9] For every $\lambda \geq 0$ one has the optimal bounds

$$\left|E_{\xi,\beta}(-\lambda t^{\xi})\right| \leq \frac{C}{1 + |\lambda t^{\xi}|} \leq C,\ t \geq 0,\ b \geq 0,$$

$$\lambda t^{\xi}\left|E_{\xi,\beta}(-\lambda t^{\xi})\right| \leq C,\ 0 < \xi < 2,\ \beta \in \mathbb{C}.$$

Results and discussion

This section summarizes the key findings of this article.

**Theorem 2.1.** Let $\varphi(x) \in C[0,1], \varphi'(x) \in L_2(0,1)$, then the unique solution of problem (1.1) – (1.3) is the function $u(x,t) \in C(\bar{\Omega})$, which has the form

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k X_k(x) E_{\alpha+\beta,1}(-\lambda_k t^{\alpha+\beta}), \tag{2.1}$$

where

$$\varphi_k = \sqrt{2}\int_0^1 \varphi(x)\sin(k\pi x)$$

and $E_{\alpha,\beta}(-z)$ is the two-parameter Mittag-Leffler function.

*Proof.* In view of the method separation of variables, any solution of problem (1.1)-(1.3) can be represented as

$$u(x,t) = \sum_{k=1}^{\infty} X_k(x)T_k(t), \quad (x,t) \in (0,1)\times(0,T), \tag{2.2}$$

and the function $\varphi(x)$ given in the following form

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k X_k(x), \quad x \in (0,1),$$

where $\varphi_k$ defined by

$$\varphi_k = \sqrt{2}\int_0^1 \varphi(x)X_k(x).$$

By substituting (2.2) into the equations (1.1)-(1.3), we have a separate problem for the variable $t$

$$\left(D_{0+}^{\beta}T_k\right)(t) + \lambda_k(I_{0+}^{\alpha}T_k)(t) = 0,\ t > 0 \tag{2.3}$$

and respect to $x$

$$X_k''(x) + \lambda_k X_k(x) = 0, \tag{2.4}$$

$$X_k(0) = X_k(1) = 0\ . \tag{2.5}$$

It is well-known the orthonormal eigenfunctions and related eigenvalues of the Dirichlet problem (2.4)-(2.5) are given by $X_k(x) = \sin(k\pi x)$ and $\lambda_k = (k\pi)^2$, respectively.

Applying $I_{0+}^{\beta}$ to equation (2.3), we have

$$I_{0+}^{\beta}\left[\left(D_{0+}^{\beta}T_k\right)(t) + \lambda_k(I_{0+}^{\alpha}T_k)(t)\right] = 0.$$

Using Lemma 1.1 and Lemma 1.2 we obtain the following equation

$$\lambda_k\left(I_{0+}^{\alpha+\beta}T_k\right)(t) + T_k(t) = T_k(0),\ t > 0.$$

The integral equation has a unique solution (se [3], P.231)

$$T_k(t) = T_k(0)E_{\alpha+\beta,1}\left(-\lambda_k t^{\alpha+\beta}\right). \tag{2.6}$$

Consequently, we obtain the solution the problem (1.1)-(1.2)

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k X_k(x)E_{\alpha+\beta,1}(-\lambda_k t^{\alpha+\beta}), \quad (x,t) \in (0,1)\times(0,T). \tag{2.7}$$

Next, we consider the function (2.7), where
$0 < \alpha < \beta \leq 1, \varphi_k = (\varphi, X_k), X_k(x) = \sqrt{2}\sin\sqrt{\lambda_k}x \cdot \lambda_k = (k\pi)^2.$

At this stage, we should prove that $u(x,t) \in C(\bar{\Omega})$ for
$\Omega = \{(x,t): 0 < x < 1, 0 < t < T\}$. For this, we have to show the uniform convergence of series (2.7) in a closed domain $\bar{\Omega}$. Now, let us estimate the coefficients $\varphi_k$. By definition

$$\varphi_k = (\varphi, X_k) = \sqrt{2}\int_0^1 \varphi(x)\sin(k\pi x)dx\,. \tag{2.8}$$

Integrating by parts the integral (2.8), we obtain

$$\varphi_k = \sqrt{2}\int_0^1 \varphi(x)d\left[-\frac{\cos(k\pi x)}{k\pi}\right]$$

$$= -\varphi(x)\frac{\sqrt{2}\cos(k\pi x)}{k\pi}\bigg|_{x=0}^{x=1} + \frac{\sqrt{2}}{k\pi}\int_0^1 \varphi'(x)\cos(k\pi x)dx.$$

If the conditions $\varphi(0) = \varphi(1) = 0$ are holds true, then it yields that

$$\varphi_k = \frac{1}{k\pi}\varphi_k^{(1)}, \tag{2.9}$$

where the function $\varphi_k^{(1)}$ defined by

$$\varphi_k^{(1)} = \int_0^1 \sqrt{2}\varphi'(x)\cos(k\pi x)\,dx. \tag{2.10}$$

In view of (2.9) and Lemma 1.3, also from the inequality $|X_k(x)| \leq C$, we get

$$|u(x,t)| \leq C\sum_{k=1}^{\infty}|\varphi_k| = C\sum_{k=1}^{\infty}\frac{1}{k}\left|\varphi_k^{(1)}\right|.$$

Therefore, we investigate the convergence of the series $\sum_{k=1}^{\infty}\frac{1}{k}\left|\varphi_k^{(1)}\right|$.

Using the Cauchy-Schwarz inequality, we have

$$\sum_{k=1}^{\infty}\frac{1}{k}\left|\varphi_k^{(1)}\right| \leq \sqrt{\sum_{k=1}^{\infty}\frac{1}{k^2}}\sqrt{\sum_{k=1}^{\infty}\left|\varphi_k^{(1)}\right|^2}.$$

Moreover, we also know that the system $Y_k(x) = \{\sqrt{2}\cos(k\pi x)\}_{k=1}^{\infty}$ is orthonormal in space $L_2(0,1)$ and for any function $g(x) \in L_2(0,1)$ the Bessel inequality holds

$$\sum_{k=1}^{\infty} |\varphi_k|^2 \leq \| \varphi(x) \|_{L_2}^2 = \int_0^1 \varphi^2(x)dx,$$

where

$$\varphi_k = (\varphi, Y_k) = \sqrt{2}\int_0^1 \varphi(x)\cos(k\pi x)\,dx.$$

So, if

$$\varphi(x) \in L_2(0,1) \Leftrightarrow \int_0^1 \varphi^2(x)dx < \infty,$$

then

$$\sum_{k=1}^{\infty} |\varphi_k|^2 < \infty,$$

i.e. the series converges.

Further, if $\varphi'(x) \in L_2(0,1)$, then for coefficients $\varphi_k^{(1)}$ of equality (2.10) using Bessel's inequality, we conclude $\sum_{k=1}^{\infty}\frac{1}{k}\left|\varphi_k^{(1)}\right|$.

Thus, if $\varphi'(x) \in L_2(0,1)$, then the number series $\sum_{k=1}^{\infty}\frac{1}{k}\left|\varphi_k^{(1)}\right|$ converges, when the next conditions hold true

$$\varphi(x) \in C[0,1], \varphi'(x) \in L_2[0,1],\ \varphi(0) = \varphi(1) = 0. \tag{2.11}$$

Consequently, the series (2.8) converges uniformly in the closed region $\bar{\Omega}$.

Therefore, the sum of this series, i.e. the function $u(x,t)$ of equality (2.1) belongs to class $C(\bar{\Omega})$.

Now let us show that the solution is unique. Assume that $u_1(x,t)$ and $u_2(x,t)$ are two solutions to the problem (1.1)-(1.3). We choose $u(x,t) = u_1(x,t) - u_2(x,t)$ so that $u(x,t)$ satisfies the equation and the initial and boundary conditions (1.2)-(1.3). Consider the following identity

$$T_k(t) = \int_0^1 u(x,t)\sin(k\pi x)dx,\ k \in N,\ t \geq 0\,. \tag{2.12}$$

Noting (2.3), we apply the operator $D_{0+}^{\beta}$ to the left side of the equation (2.12)

$$\begin{aligned}\left(D_{0+}^{\beta}T_k\right)(t) &= \int_0^1 \left(D_{0+}^{\beta}u\right)(x,t)\sin(k\pi x)dx \\ &= -(k\pi)^2 I_{0+}^{\alpha}\int_0^1 u(x,t)\sin(k\pi x)dx \\ &= -(k\pi)^2(I_{0+}^{\alpha}T_k)(t),\ k \in N,\ t \geq 0.\end{aligned}$$

As a result of (1.2) and (1.3) we have

$$T_k(0) = \int_0^1 u(x,0)\sin(k\pi x)dx = \int_0^1 \varphi(x)\sin(k\pi x)dx = 0.$$

In view of (2.6) we deduce that

$$T_k(t) = T_k(0)E_{\alpha+\beta,1}\left(-\lambda_k t^{\alpha+\beta}\right) = 0.$$

Since $T_k(0) = 0$, which means $u(x,t) = 0$. Hence $u_1(x,t) = u_2(x,t)$, and the problem (1.1)-(1.3) has a unique solution.

By applying the operators $D_{0+}^{\beta}$ and $I_{0+}^{\alpha}$ to the identity (2.7), we get

$$(D_{0+}^{\beta}u)(x,t) = D_{0+}^{\beta}[\sum_{k=1}^{\infty}\varphi_k X_k(x)E_{\alpha+\beta,1}(-\lambda_k t^{\alpha+\beta})]$$
$$= \sum_{k=1}^{\infty}\varphi_k X_k(x)D_{0+}^{\beta}[E_{\alpha+\beta,1}(-\lambda_k t^{\alpha+\beta})] \quad (2.13)$$
$$= -\sum_{k=1}^{\infty}\varphi_k X_k(x)\lambda_k t^{\alpha}E_{\alpha+\beta,\alpha+1}(-\lambda_k t^{\alpha+\beta})$$

and

$$(I_{0+}^{\alpha}u)(x,t) = I_{0+}^{\alpha}[\sum_{k=1}^{\infty}\varphi_k X_k(x)E_{\alpha+\beta,1}(-\lambda_k t^{\alpha+\beta})]$$
$$= \sum_{k=1}^{\infty}\varphi_k X_k(x)I_{0+}^{\alpha}[E_{\alpha+\beta,1}(-\lambda_k t^{\alpha+\beta})] \quad (2.14)$$
$$= -\sum_{k=1}^{\infty}\varphi_k X_k(x)t^{\alpha}E_{\alpha+\beta,\alpha+1}(-\lambda_k t^{\alpha+\beta})$$

.

By using the operator $\frac{\partial^2}{\partial x^2}$ to the (2.14), we deduce that

$$\frac{\partial^2}{\partial x^2}(I_{0+}^{\alpha}u)(x,t) = \sum_{k=1}^{\infty}\varphi_k \frac{\partial^2}{\partial x^2}X_k(x)t^{\alpha}E_{\alpha+\beta,\alpha+1}(-\lambda_k t^{\alpha+\beta})$$
$$= \sum_{k=1}^{\infty}\varphi_k X_k(x)\lambda_k t^{\alpha}E_{\alpha+\beta,\alpha+1}(-\lambda_k t^{\alpha+\beta}).$$

Next, we show that $D_{0+}^{\beta} \in C(\Omega)$ and $I_{0+}^{\alpha} \in C(\Omega)$.

Let $\delta$ be an arbitrary, sufficiently small positive number. Then for all $0 < \delta \le t$, from Lemma 1.3 , we get

$$\left|D_{0+}^{\beta}u(x,t)\right| = \left|\sum_{k=1}^{\infty} \varphi_k X_k(x)\lambda_k t^{\alpha}E_{\alpha+\beta,\alpha+1}(-\lambda_k t^{\alpha+\beta})\right|$$
$$= \left|\sum_{k=1}^{\infty} \varphi_k X_k(x)t^{-\beta}\lambda_k t^{\alpha+\beta}E_{\alpha+\beta,\alpha+1}(-\lambda_k t^{\alpha+\beta})\right|$$
$$\le C\sum_{k=1}^{\infty} |\varphi_k|$$

and

$$|I_{0+}^{\alpha}u(x,t)| = \left|\sum_{k=1}^{\infty} \varphi_k X_k(x)\lambda_k t^{\alpha}E_{\alpha+\beta,\alpha+1}(-\lambda_k t^{\alpha+\beta})\right|$$
$$= \left|\sum_{k=1}^{\infty} \varphi_k X_k(x)t^{-\beta}\lambda_k t^{\alpha+\beta}E_{\alpha+\beta,\alpha+1}(-\lambda_k t^{\alpha+\beta})\right|$$
$$\le C\sum_{k=1}^{\infty} |\varphi_k|.$$

If the conditions (2.11) hold true, the series $\sum_{k=1}^{\infty}|\varphi_k|$ converges, and then the series (2.13) and (2.14) representing the function $D_{0+}^{\beta}u(x,t)$ and $\frac{\partial^2}{\partial x^2}(I_{0+}^{\alpha}u)(x,t)$ converges uniformly in any closed subdomain $\bar{\Omega}_{\delta}$ of the domain $\Omega$. Therefore, due to the arbitrariness of the number $\delta$, we have $D_{0+}^{\beta}u \in C(\Omega)$ and $I_{0+}^{\alpha}u \in C(\Omega)$.

**Conclusion**

In this paper, the main results include the presentation of well-known properties associated with fractional operators and the establishment of a classical solution to this problem. The key conclusions are summarized using a theorem that provides an explicit form of the solution. The solution is expressed as a series including the two-parameter Mittag-Leffler function and orthonormal eigenfunctions of the Sturm-Liouville operator. The uniqueness of the solution is proved, which guarantees that the problem has a unique solution.

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## УАҚЫТ БОЙЫНША БЕЙЛОКАЛДЫ ДИФФУЗИЯ ТЕҢДЕУІ ҮШІН БАСТАПҚЫ-ШЕТТІК ЕСЕП

**Аңдатпа**

Бұл мақалада Капуто мағынасындағы бөлшек ретті туынды мен Риман-Лиувилл мағынасындағы бөлшек ретті интегралдау операторлары қатысқан бөлшек ретті диффузия теңдеуі қарастырылады. Теңдеу кеңістік бойынша 0<x<1 кесіндісінде және уақыт бойынша 0<t<T кесіндісінде анықталған аймағында бастапқы және шекаралық шарттармен толықтырылған. Бөлшек операторлар 0<α<β≤1 арқылы, яғни β ретті Капутоның бөлшек ретті туындысы және α ретті Риман-Лиувилл бөлшек ретті интегралы арқылы анықталады. Негізгі нәтижелер – бөлшек операторлармен байланысты белгілі қасиеттерді ұсыну мен жалғыз шешімнің болуы. Негізгі тұжырымдар шешімнің айқын формасын қамтамасыз ететін теорема арқылы жалпыланған. Шешім екі параметрлі Миттаг-Леффлер функциясын және Штурм-Лиувилл операторының ортонормалды меншікті функцияларын қамтитын қатар түрінде көрсетіледі. Шешімнің жинақтылығы дәлелденді, бұл есептің бастапқы функция үшін белгілі бір жағдайларда жалғыз, нақты анықталған шешімі болуын қамтамасыз етеді. Сонымен қатар мақалада жинақтылықты талдау үшін, Миттаг-Леффлер функциясымен байланысты маңызды бағалаулар енгізіліп, дәлелденеді. Қатардың жинақтылығы зерттеледі және шешімнің белгілі бір функционалды кеңістікке жату шарттары белгіленеді. Бұл есептің шеңберінде оның ерекшелігін көрсететін жалғыз шешім көрсетіледі. Көрсетілген аймақтағы шешімнің үздіксіздігі қатардың біркелкі жинақты болуымен дәлелденді.

**Тірек сөздер:** бөлшек туынды, интегралдық теңдеу, айнымалыларды ажырату әдісі, уақыт бойынша бейлокалды диффузия теңдеуі.

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## НАЧАЛЬНО-КРАЕВЫЕ ЗАДАЧИ ДЛЯ УРАВНЕНИЯ НЕЛОКАЛЬНОЙ ПО ВРЕМЕНИ ДИФФУЗИИ

**Аннотация**

В этой статье исследуется уравнение дробной диффузии, включающее дробную производную Капуто и дробный интеграл Римана-Лиувилля. Уравнение дополнено начальными и граничными условиями в области, определяемой интервалом 0<x<1 по пространственной переменной и 0<t<T по временной перемен-

ной. Дробные операторы определены строго, используя дробную производную Капуто порядка β и дробный интеграл Римана-Лиувилля порядка α, где 0<α<β≤1. Основные результаты включают представление хорошо известных свойств, связанных с дробными операторами, и установлено единственное решение данной задачи. Ключевые выводы обобщены с помощью теоремы, которая обеспечивает явную форму решения. Решение выражается в виде ряда, включающего двухпараметрическую функцию Миттага-Леффлера и ортонормированные собственные функции оператора Штурма-Лиувилля. Доказана единственность решения, гарантирующая, что задача имеет единственное, четко определенное решение при определенных условиях для исходной функции. Кроме того, в статье вводятся и доказываются оценки, связанные с функцией Миттага-Леффлера, предоставляя оценки, имеющие решающее значение для анализа сходимости. Исследуется сходимость ряда и устанавливаются условия принадлежности решения определенному функциональному пространству. Демонстрируется единственное решение, подчеркивающее его необычность в рамках данной задачи. Непрерывность решения в указанной области подтверждается равномерной сходимостью ряда.

**Ключевые слова:** дробная производная, интегральное уравнение, метод разделения переменных, уравнение диффузии бейлокала по времени.